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(Dedicated to the memory of Professor K. L. Singh)

SOME VARIANTS OF VIGNOLI'S THEOREM ON α-CONTRACTION

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1. Introduction. The following theorem of *A*. Vignoli [1] may be regarded as a nice extension of the well-known Darbo's fixed point theorem in the case of a ball. Throughout the paper we assume X to be a real Banach space, $B(0,R)$ a ball with centre at 0 and radius R and $\delta B(0, R)$ the boundary of the ball $B(0,R)$.

Theorem 1. Let $T : B(0, R) \to X$ be α-contractive with the constant $k, (0 \leqslant k < 1)$ and let T satisfies the following condition on the boundary of $B(0, R)$:

(a) if $Tx = \beta x$ for some $x \in \delta B(0,R)$ then $\beta \leqslant \mu$ where μ is any real number satisfying the inequality

(b) $0 \leqslant k < 1 - | 1 - \mu |$.

Then there exists $x \in B(0,R)$ such that $T(x) = \mu x$.

Remark 1. When $\mu = 1$, Vignoli's theorem reduces to an extension of Darbo's fixed point theorem in the case of a ball, just like S. Reich's [2] extension of Brouwer's fixod point theorem.

As D. K. Bayen and S. K. Chatterjea [3] considered a nice variant of Reich's theorem and A. K. Sarkar and S. K. Chatterjea [4] have recently considered some other variants of Vignoli's theorem for $\mu = 1$, we intend to consider some new variants of Vignoli's theorem for any real number μ subject to a certain condition.

2. Some new variants of Vignoli's theorem

Theorem 2. Let $T : B(0,R) \to X$ be α-contractive with constant k, $0 \leqslant k < 1$ and T satisfy the following conditions.

(a) if $Tx = \beta x$ for some $x \in \delta B(0,R)$ then $\beta \geqslant \mu$ where μ is any real number satisfying the inequality

(b) $0 \leqslant k + | 1+\mu | < 1$.

Then there exists $z \in B(0,R)$ such that $T(z) = -\mu z$.

Proof. We consider the mapping $F : B(0,R) \to X$ defined by $F(x) = Tx + (1+\mu)x$ for $x \in B(0,R)$ and μ satisfies the condition (b) of the theorem. The mapping F is a α-contractive. Indeed, let A be any subset of $B(0,R)$. Then we have

$$\begin{aligned}\alpha(F(A)) &\leqslant \alpha(T(A)) + | 1+\mu | \alpha(A) \\ &\leqslant k\,\alpha(A) + | 1+\mu | \alpha(A) \\ &= (k + | 1+\mu |)\,\alpha(A)\end{aligned}$$

condition (b) of the theorem implies $0 \leqslant k + | 1+\mu | < 1$, so that F is α-contractive with constant $k + | 1+\mu |$.

Let $r : X \to B(0,R)$ be a radial retraction. Since r is non-expansive, the composite mapping $roF : B(0,R) \to B(0,R)$ is α-contractive. Thus by Darbo's fixed point theorem there exists $z \in B(0,R)$ such that $z = roF(z)$.

Thus $$z = roF(z) = \begin{cases} F(z), & \text{if } \| F(z) \| \leqslant R \\ R\dfrac{F(z)}{\| F(z) \|} & \text{if } \| F(z) \| > R. \end{cases}$$

We shall now show that z is the fixed point of F, i. e. $F(z) = z$. If $z \in \operatorname{Int} B(0,R)$ then $F(z) = z$, otherwise $z = R\dfrac{F(z)}{\| F(z) \|}$, i. e.

$F(z) = \dfrac{\| F(z) \|}{R} z$, would imply that $\| z \| = R$ which contradicts the assumption $\| z \| < R$. Next suppose that $z \in \delta B(0,R)$ and z is not a fixed point of F. Then $\| F(z) \| > R$ and

$$z = R \frac{F(z)}{\| F(z) \|}$$

$$\Rightarrow F(z) = \frac{\| F(z) \|}{R} z$$

$$\Rightarrow T(z) + (1+\mu)z = \frac{\| F(z) \|}{R} z$$

$$\Rightarrow T(z) = \left(\frac{\| F(z) \|}{R} - 1 - \mu \right) z$$

$$= \beta z, \text{ where } \beta = \frac{\| F(z) \|}{R} - 1 - \mu$$

Now $\beta = \frac{\| F(z) \|}{R} - 1 - \mu$

$\Rightarrow -\beta = \mu + \left(1 - \frac{\| F(z) \|}{R}\right) < \mu$, which contradicts the condition (a) of the theorem. Thus $F(z) = z$ *i. e.* z is a fixed point of F. Hence $F(z) = z$ implies $T(z) = -\mu z$.

Remark 2. When $\mu = -1$, theorem 2 reduces to another extension (b) of the theorem implies $\mu \in (-2,0)$.

Theorem 3. Let $T \cdot B(0,R) \to X$ be α-contractive with the constant k, $0 \leqslant k < 1$ and let T satisfy the following conditions:

(a) $Tx = \beta x$ for some $x \in \text{Int } B(0,R)$ then $\beta \geqslant -\mu$ where μ is any real number satisfying the inequality

(b) $0 \leqslant k + | 1 + \mu | < 1$.

Then there exists $z \in B(0,R)$ such that $T(z) = -\mu z$

Proof. Proceeding exactly in the same way as in the proof of theorem 2, we can prove that

$$z = roF(z) = \begin{cases} F(z), \text{ if } \| F(z) \| \leqslant R \\ R \dfrac{F(z)}{\| F(z) \|} \text{ if } \| F(z) \| > R. \end{cases}$$

We shall now show that z is a fixed point of F *i.e.* $z = F(z)$.

If $z \in \delta B(0,R)$ then two cases may arise.

(i) $\| F(z) \| \leqslant R$ and

(ii) $\| F(z) \| > R$.

In the case (1) $z = F(z)$ *i.e.* z is the fixed point of F. But in the case (ii) we have $z = R \dfrac{F(z)}{\| F(z) \|}$ which would imply $F(z) = \dfrac{\| F(z) \| z}{R}$

and no conclusion regarding the fixed point of F can be drawn.

If $z \in Int\ B(0,R)$ then $z = F(z)$, otherwise $z = R \dfrac{F(z)}{\| F(z) \|}$ which would imply $F(z) = \dfrac{\| F(z) \|}{R} z$

$$\Rightarrow (1+\mu+\beta)z = \frac{\| (1+\mu+\beta)z \|}{R}$$
$$= \frac{|(1+\mu+\beta)| \, \| z \|}{R}$$

$\Rightarrow z \dfrac{\| z \|}{R} z$, which contradicts the assumption $\| z \| < R$.

Therefore $F(z) - z$ and hence $T(z) - - \mu z$.

Remark 3. When $\mu = -1$, theorem 3 reduces to another extension of Darbo's fixed point theorem in the case of a ball like theorem 2. Also condition (b) of theorem 3 implies that $\mu \in (-2,0)$.

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