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*(Dedicated to the memory of Professor K. L. Singh)*

# RIGHT ALTERNATIVE RINGS WITH COMMUTATORS IN THE MIDDLE NUCLEUS

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## ABSTRACT

Let $R$ be a right alternative ring with characteristic $\neq 2$ and with commutators in the middle nucleus. We show that if $R$ is left primitive, then it is either associative or simple with right identity element. As a by-product we prove that if $R$ is simple, not associative then it has no proper left ideals generated by all alternators

## 1. INTRODUCTION

Using the standard associator notation $(x,y,z) = (xy)z-x(yz)$, a right alternative ring is a non-associative ring which satisfies the identity

(i) $(x,y,z) = 0.$

Also, it is known [3] that a right alternative ring with characteristic $\neq 2$ satisfies the following identity :

(1') $(x,y,z) + (x,z,y) = 0.$

In a right alternative ring $R$ the middle nucleus $M$ is defined as the set

$$M = \{z \in R : (x,y,z) = 0 \text{ for all } x,y \in R\}.$$

Throughout this paper, we assume $R$ to be a right alternative ring with characteristic $\neq 2$ such that

(2) $(x,y,(z,w)) = 0$

for all $x,y,z,w$ in $R$ where the commutator $(z,w) = zw-wz$.

That is, $(R,R) \subseteq M$.

A straightforward verification shows that any ring satisfies

(T) $(wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z$

which is known as the Teichmüller identity.

## 2. CONSTRUCTION OF IDEALS

**Lemma .** For any ring $R$, $(R,R,R) + R(R,R,R)$ is a two-sided ideal of $R$.

This is proved in [2].

An alternator is an associator of the form $(x,x,y)$, $(x,y,x)$, or $(y,x,x)$.

**Lemma 2.** Let $A$ be a left ideal of $R$ generated by all alternators. Then

(a) $S = \{s \in A: sR \subseteq A\}$ is a two-sided ideal of $R$.

(b) $(R,R,A) \subseteq S$.

**Proof.** (a) This is Theorem 1(a) of [1]

(b) Let $w,x,z \in R$ and $a \in A$. Using (T)

$$(w,x,a)z = (wx,a,z) - (w,xa,z) + (w,x,az) - w(x,a,z)$$

$$= -(wx,z,a) + (w,z,xa) + (w,x,za) - w(x,z,a) \in A$$

(Using (1′) and (2)). Since $(R,R,A) \subseteq A$, we have $(R,R\ A) \subseteq S$.

**Theorem 1.** If $R$ has a maximal left ideal $A \neq (0)$ generated by all alternators which contains no two sided ideal of $R$ other than (0), then $R$ is associative.

**Proof.** By Lemma 2, $S$ is a two-sided ideal of $R$ contained in $A$. Hence $S = (0)$. Since $(R,R,A) \subseteq S$, we have $(R,R,A) = 0$.

$A+AR$ is clearly a left ideal by *Eq.* (1′). Since $A$ contains all alternators, $(A,R,R) \subset A$ and we see that $A+AR$ is a right ideal as well. Thus $A+AR$ is a two-sided ided of $R$. Since $A \subset A+AR$, we must have $A+AR = R$. Hence $\neq$

$$(R,R,R) = (R,R,A+AR)$$
$$\subseteq (R,R,A) + (R,R,AR)$$
$$\subseteq (R,R,A) + (R,R,RA) \quad \text{(using (2))}$$
$$\subseteq (R,R,A) = (0).$$

Hence $R$ is associative.

**Theorem 2.** If $R$ is a simple and not associative, then $R$ has no proper left ideal generated by all alternators.

**Proof.** Suppose, for example that $R$ has a left ideal $A \neq (0)$ generated by all alternators. We shall show that $A = R$. Now as in the proof of Theorem 1, $A+AR$ is a two-sided ideal of $R$ and $(0) \neq A \subseteq A+AR$ implies that $A+AR \neq (0)$ and since $R$ is a simple ring, we have $A+AR = R$. Then from (2), we have

$$(R,R,R) = (R,R,A=AR) = (R,R,A) + (R,R,AR)$$
$$\subseteq A+(R,R,RA) \subseteq A.$$

Now by Lemma 1, $(R,R,R) + R(R,R,R)$ is a two-sided of $R$. Also since $R$ is not associative we have that

$$(R,R,R) \neq (0) \text{ and } (0) \neq (R,R,R) \subseteq (R,R,R) + R(R,R,R)$$

implies that $(R,R,R) + R(R,R,R)$ is a non-zero ideal of the simple ring $R$ and, therefore, $(R,R,R) + (R,R,R) = R$. Thus

$$R = (R,R,R) + R(R,R,R) \subseteq A+RA \subseteq A.$$

Therefore $A = R$.

## 3. LEFT PRIMITIVE RINGS

**Definition 1.** A left ideal $A$ of $R$ is clled regular if there exists an element $g \in R$ such that $x-xg \in A$ for all $x \in R$.

**Definition 2.** A ring $R$ is called left primitive if it contains a regular maximal left ideal which contains no two sided of $R$ other then (0).

**Theorem 3.** If $R$ is a left primitive ring, then either $R$ is associative or it is simple with a right identity element.

**Proof.** Let $A$ be a regular maximal left ideal generated by all alternators which contains no two sided ideal of $R$ other than (0). If $A \neq (0)$, then by Theorem 1, $R$ is associative Thus assume that (0) is maximal implies, using Zorn's Lemma, that $R$ has no proper ideal. Therefore $R$ is simple. Since (0) is regular, there exists $g \in R$ such that $x-xg \in (0)$ for all $x \in R$. That is, $x=xg$ for all $x \in R$ and $g$ is a right identity element. Hence $R$ is simple with right identity element.

## REFERENCES


[1] I. R. Hentzel, The alternator ideal of right alternative rings, *Bulletin of the Iranian Mathematical Society* 16 (1981), 15-20.

[2] E. Kleinfeld, Assosymmetric rings, *Proc. Amer. Math. Soc.* 88 (1957), 983-986.

[3] A. Thedy, Right alternative rings, *J. Algebra* 37 (1975), 1-43.