Jñānābha, Vol. 22, 1992

(Dedicated to the memory of Professor K. L. Singh)

FIXED POINTS FOR NONEXPANSIVE MULTIVALUED MAPPING AND THE OPIAL'S CONDITION

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(Received: June 5, 1992)

ABSTRACT

Before Professor K L Singh's death some fixed point theorems for multivalued nonexpansive mappings on weakly compact and convex subsets of opial spaces, were discussed by both of us. In the present paper, these are reorganised.

INTRODUCTION

Several fixed point theorems have been obtained for nonexpansive mappings [1], [2], [3]. Fixed point theorems in lacally convex topological vector spaces for multivalued nonexpansive mappings have been studied [4], [10], [11], [12] and others. In this paper we shall continue these investigations.

Throughout the paper, we shall use E to denote a Hausdroff locally convex topological vector space, and F to denote the family of continuous seminorms generating the topology of E. Also $C(E)$ will denote the family of nonempty compact subsets of E.

For each $p \in F$ and $A, B \in C(E)$, we define

$\delta_p(A,B) = sup\{p(a-b): a \in A, b \in B\}$, and

$$D_p(A, B) = max\left\{\sup_{a \in A} \inf_{b \in B} [p(a-b)], \sup_{b \in B} \inf_{a \in A} [p(a-b)]\right\}.$$

Although p is only a seminorm, D_p is a Hausdroff metric on $C(E)$ [4].

Definition 1: Let K be a nonempty subset of E a mapping $T: K \to C(E)$ is called a multivalued contraction if there exists a constant k_p, $0 \leq k_p < 1$, such that for each $x, y \in K$, $D_p(Tx, Ty) \leq k_p p(x-y)$.

AMS (1980) *Subject Classification* Primary 47H10

*Died on November 22, 1990

T is called multivalued nonexpansive if for each $x, y \in K$, $D_p(Tx, Ty) \leqq p(x-y)$.

Definition 2: A set C in a topological vector space E is called convex if $\alpha x + (1-\alpha) y \in C$ whenever $x, y \in C$ and $0 \leqq \alpha \leqq 1$.

For any $x, y \in E$, we set

$(x, y] = \{(1 - \alpha) x + \alpha y: 0 < \alpha \leqq 1\}$, and

$[x, y] = \{(1 - \alpha) x + \alpha y: 0 \leqslant \alpha \leqq 1\}$.

Definition 3: E is said to satisfy the Opial's condition if for each $x \in E$ and every net $\{x_\alpha\}$ converging weakly to x, then for each $p \in F$, we have

$lim\, p(x_\alpha - y) > lim\, p(x_\alpha - x)$ for $y \neq x$.

Definition 4: Let D be any nonempty subet of E, a mapping $T{:}D \rightarrow C(E)$ is said to satisfy the boundary condition (α) if for all $x \in D$ and all $y \in Tx$, $(x, y] \cap \neq D\, \phi$.

The following variant of Nadler's result [8] will be used.

Lemma. If $A, B \in C(E)$, then for each $a \in A$, there is a $b \in B$ such that $p(a - b) \leqq D_p(A, B)$ for all $p \in F$.

Theorem 1. Let K be a nonempty, weakly, compact and convex subset of E, and $T{:}K \rightarrow C(E)$ be nonexpansive. If E satisfies the Opial's condition, then $G_r(I - T)$ is closed in $E_w \times E$, where E_w is E with its weak topology and I is the identity mapping.

Proof: Let $\{(x_\alpha, y_\alpha)\}$ be a net in $G_r(I - T)$ such that $x_\alpha \rightarrow x$ weakly and $y_\alpha \rightarrow y$. It is not difficult to see that $x \in K$, so we must show that $y \in (I - T)(x)$. Now, for every α, there exists an $v_\alpha \in Tx_\alpha$ such that $y_\alpha = x_\alpha - v_\alpha$, by the Lemma, we can find an $v'_\alpha \in Tx$ such that $p(v_\alpha - v'_\alpha) \leqq D_p(Tx_\alpha, Tx)$. Thus we have

$$p(v_\alpha - v'_\alpha) \leqq D_r(Tx_\alpha, Tx) \leqq p(x_\alpha - x)$$

because T is nonexpansive. Therefore

$$\lim p(x_\alpha - x) \geqq \lim p(v_\alpha - v'_\alpha) = \lim p(x_\alpha - y_\alpha - v'_\alpha).$$

By compactness of Tx, there is a convergent subnet in Tx still denoted by $\{v'_\alpha\}$ such that $v'_\alpha \to v$ for some $v \in Tx$. Therefore, we have

$$\lim p(x_\alpha - x) \geqq \lim p(x_\alpha - y - v).$$

Since E satisfies the Opial's condition, then we obtain $y + v = x$. That is $y = x - v \in (I - T)(x)$.

Theorem 2. Let E be an Opial's space, K a nonempty, weakly compact, and convex subset of E. $T{:}K \to C(E)$ be a multivalued non-expansive mapping which satisfies the boundary condition (α). Then T has a fixed point in K i e., there exists $x \in K$ such that $x \in Tx$.

Proof: We shall first show that $(I - T)(K)$ is closed.

Let y be a limit point of $(I - T)(K)$. Then there is a net $\{y_\alpha\}$ with $y_a \in (I - T)\, x_\alpha$ for some $x_\alpha \in K$ and $y_\alpha \to y$. This implies that $x_\alpha - y_\alpha \in Tx_\alpha$ and $y_\alpha \to y$.

By the compactness of K, we know that there exists an $w \in K$ and a subnet $\{x_{\alpha_i}\}$ of $\{x_\alpha\}$ such that $x_{\alpha_i} \to w$ weakly. Therefore, there exists a subnet $\{y_{\alpha_i}\} \subseteq \{y_\alpha\}$ such that $y_{\alpha_i} \to y$, and by the Theorem 1, $y \in (I - T)(w)$. Thus, $(I - T)(K)$ is closed.

Let $z \in K$ be fixed and let $r_n = 1 - 1/n$ for each n. Define $T_n{:}K \to C(E)$ by $T_n(x) = r_n Tx + (1 - r_n)\, z$ for $x \in K$ Then $T_n x \in C(E)$ and T_n satisfies the boundary condition (α), since so does T. Each T_n is contraction map, it follows from a variant of Theorem 2 [6], there exists an $x_n \in T_n x_n$. Therefore, $x_n \in r_n T x_n + (1 - r_n)z$. This implies that there is $y_n \in Tx_n$, such that $x_n = r_n y_n + (1 - r_n)z$. Thus, $x_n - y_n = (r_n - 1)y_n + (1 - r_n)z \to 0$ as $n \to \infty$. Therefore, we shall have $0 \in (I - T)(K)$ because it is closed. Hence, there exists an $x \in K$ such that $x \in Tx$.

Corollary Let E be an Opial's space, K a nonempty, compact, and convex subset of E and $T:K\to C(E)$ be a multivalued nonexpansive mapping which satisfies the boundary condition (α). Then T has a fixed point in K.

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