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(Dedicated to the memory of Professor K. L. Singh)

EXISTENCE AND UNIQUENESS FOR A NONLINEAR TWO POINT BOUNDARY VALUE PROBLFM

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ABSTRACT

In this paper we study a nonlinear two point boundary value problem. We prove an uniqueness theorem, an existence theorem of constructive type, and an existence and uniqueness theorem of the solution of the two point boundary value problem, using iteration processes and fixed point methods in a Hilbert space.

1. INTRODUCTION

This paper concerns the following nonlinear two point boundary value problem:

Find $u(t) \in H^2[0,1]$ such that

$$(1.1) \quad \begin{cases} u''(t) + \lambda f(u(t)) = 0 \\ u(0) = u(1) = 0, \end{cases}$$

where $H^2[0,1]$ is the completion of $C^2[0,1]$ with respect to the L^2 norm. Our object is to obtain an uniqueness theorem, an exi tence theorem of constructive type and, as corollary, an existence and uniqueness theorem of the solution of problem (1.1) for λ fixed in some range. For simplicity, we indicate the main result of this paper, assuming f to be a twice differentiable function. If f satisfies these conditions

(i) $f(t) > 0$

(ii) $0 \leqslant f'(t) \leqslant L$

(iii) $f''(t) < 0$

for some $L > 0$ and for all $t \geqslant 0$ then problem (1.1) has an unique solution for

$$0 < \lambda < \frac{\pi^2}{L}, \lambda \neq \frac{1}{LK}$$

where $K^2 = \int_0^1 \int_0^1 G^2(x,t)\, dtdx = \frac{1}{90}$ and $G(x,t)$ is the triangular kernel. Moreover the solution is given by an iterative sequence. The existence result is obtained from the following fixed point theorem. Let H a Hilbert space and $T: H \to H$ an operator. If T is monotone and Lipschitz continuous operator with constant $\beta > 1$ then the following sequence

$$\begin{cases} x_0 \in X \\ x_{n+1} = x_n + \mu(Tx_n - x_n), \end{cases}$$

where $0 < \mu < \frac{2}{\beta - 1}$, converges to the unique fixed point of T.

The paper is organized as follows. Section 2 is devoted to the introduction of some basic results on a two point boundary value problem. Section 3 contains the proof of a fixed point theorem in Hilbert spaces. Finally in section 4 we prove an uniqueness theorem of the solution, an existence theorem of the solution of constructive type, and as corollary, an existence and uniqueness theorem, for a class of nonlinear two point boundary value problems.

2. Preliminary facts. In this section we assume that $f: R \to R^+$ is a positive and locally Lipschitz continuous function.

Proposition 2.1 *All solutions of (1 1) are strictly positive on* $(0,1)$ *for* $\lambda > 0$.

Proof. Since f and λ are both positive, then we must have $u''(x) < 0$ for all $x \in (0,1)$. Taylor's expansion of solution $u(x)$ near $x = 0$ gives us

$$u(x) = xu'(0) + \frac{1}{2} x^2 u''(\xi_x),\ 0 < \xi_x < 1.$$

Letting $x = 1$, we deduce that

$$u(1) = u'(0) + \frac{1}{2} u''(\xi_1)$$

and, hence, that $u'(0) > 0$. Now we have

$$u(x) = ax + O(x),\ x \to 0\ (a > 0)$$

and therefore there exists c, $0<c<1$, such that $u(x) > 0$ on $(0,1)$. Let

$$\gamma = sup\{c: u(x) > 0\ on\ (0,c)\}.$$

If $\gamma = 1$ then our statement is proved. Assuming, by contradiction, $\gamma < 1$, necessarily it must be that $u(\gamma) = 0$. Rolle's theorem assures that there exist c_1, c_2 such that

$$0 < c_1 < \gamma < c_2 < 1$$

and

$$u'(c_1) = u'(c_2) = 0.$$

The same argument applied to function $u'(x)$ on interval (c_1, c_2), implies that there exists c_3 $c_1 < c_3 < c_2$ such that

$$u''(c_3) = 0.$$

This is a contradiction, since $u''(x) < 0$ on $(0,1)$.

Proposition 2.2 *Any solution of (1.1) has exactly one maximum on* (0,1).

Proof We prove that by contradiction, assuming that there exists a solution $u_*(x)$ having two maxima, say a and b $(a<b)$ such that

$$u_*(a) = u_*(b) = \max_{x\in[0,1]} u_*(x).$$

Rolle's theorem applied to $u'_*(x)$ on (a,b) implies that there exists a $c \in (a,b)$ such that $u''_*(c) = 0$. We have a contradiction and hence the proposition is proved.

Proposition 2.3 *Any solution of* (1.1) *is symmetric with respect to* $x = \frac{1}{2}$ *and, moreover,*

$$\max_{x\in[0,1]} u(x) = u(\tfrac{1}{2}) \geqslant \|u(x)\|_{L^2[0,1]}.$$

Proof. We claim that any solution $u(x)$ of (1.1) satisfies

$$u(1-x) = u(x).$$

It is easy to verify that function $y(x) = u(1-x)$ is a solution of (1.1) too. Indeed, we have

$$y' = -u'(1-x), \quad y'' = u'(1-x)$$

$$y(0) = u(0) = 0, \quad y(1) = u(1) = 0$$

and therefore

$$\begin{cases} y''(x) + \lambda f(y(x)) = 0 \\ y(0) = y(1) = 0. \end{cases}$$

Now we have to prove that $y(x) = u(x)$. Remark that it must be

$$y'(\tfrac{1}{2}) = u'(\tfrac{1}{2}) = 0.$$

Let $r = y(\frac{1}{2}) = u(\frac{1}{2})$. Since, f is locally Lipschitz continuous, then the initial value problem

$$\begin{cases} u'' + \lambda f(u) = 0 \\ u(\frac{1}{2}) = r \\ u'(\frac{1}{2}) = 0 \end{cases}$$

has a unique solution, then it follows that $y(x) = u(x)$.
Taking in account Propositions 2.1 and 2 2, we see that

$$\max_{x\in[0,1]} u(x) = u(\tfrac{1}{2}) \geqslant \|u(x)\|_{L^2[0,1]}.$$

Proposition 2.4 *If u_1, u_2 are distinct solutions for a fixed λ, then either $u_1(t) < u_2(t)$ for $0 < t < 1$ or $u_1(t) > u_2(t)$ for $0 \leqslant t \leqslant 1$.*

Proof Let $u_1(t)$ and $u_2(t)$ be two solutions for a fixed λ. Denoting $r_i = \max_{t\in[0,1]} u_i(t)$, $i = 1,2$ it must be that $r_1 \neq r_2$ in view of Prop, 2.3. Assume $r_1 < r_2$ and prove that $u_1(t) < u_2(t)$ for $0 < t < 1$. Integrating equation

$$u''_i(t) + \lambda f(u_i(t)) = 0$$

over $(t, \frac{1}{2})$, we obtain

$$\tfrac{1}{2} u'^2_i(t) + \lambda F(u_i(t)) = \lambda F(r_i), \; i = 1,2$$

where $F(\xi) = \int_0^\xi f(x)\,dx$. Since F is increasing $F(r_1) < F(r_2)$ and consequently

$$\tfrac{1}{2} u'^2_1(0) = \lambda F(r_1) < \lambda F(r_2) = \tfrac{1}{2} u'^2_2(0)$$

therefore we deduce

$$u'_1(0) < u'_2(0).$$

Thus we have eiher $u_1(t) < u_2(t)$ for $0 < t \leqslant \frac{1}{2}$ or there exists a smallest c, $0 < c < \frac{1}{2}$ such that $u_1(c) = u_2(c)$. Equations

$$\tfrac{1}{2} u'_i = \lambda(F(r_i) - F(u_i)), \; i = 1,2$$

become

$$u'_i = (2\lambda)^{\frac{1}{2}} (F(r) - F(\omega))^{\frac{1}{2}} \text{ ,for } 0 \leqslant t \leqslant \tfrac{1}{2},$$

since u is increasing on $0 \leqslant t \leqslant \frac{1}{2}$ Then we have

$$\int_0^{u_i(t)} \frac{d\omega}{(F(r) - F(\omega))^{1/2}} = t(2\lambda)^{1/2}.$$

Equality $u_1(c) = u_2(c)$ implies that

$$(F(r_1) - F(r_2))^{\frac{1}{2}} = (F(r_2) - F(u))^{\frac{1}{2}}$$

hence $F(r_1) = F(r_2)$. Thus it must be $r_1 = r_2$ against hypothesis $r < r_2$

3. A fixed point theorem.

We recall some prelimiary definitons.

Definition 3.1 *An operator* $T: Y \subseteq H \to H$ *is called monotone if for all* $x, y \in Y$

(3.3) $\quad (Tx - Ty, x - y) \geqslant 0.$

Definition 3.2 *An operator $T:Y \subseteq H \to H$ is called Lipschitz continuous if there exists a $\beta > 0$ such that for all $x,y \in Y$*

(3.2) $\|Tx - Ty\| \leqslant \beta\|x - y\|$.

Now we state and prove the main result of this section.

Theorem 3.3 *Let H a Hilbert space, endowed with the inner product $(,)$ and let T: $H \to H$ be a monotone and Lipschitz continuous operator with $\beta > 1$. Then the sequence*

$$\begin{cases} x_0 \in H \\ x_{n+1} = x_n + \mu(Tx_n - x_n), \end{cases}$$

where $0 < \mu < \dfrac{2}{\beta - 1}$, converges to a fixed point of T.

Proof. First of all we remark that

$$x_{n+1} = (1 - \mu)x_n + \mu Tx_n.$$

Let us prove by induction, that $x_n \in H$ for all n. By hypothesis $x_0 \in H$. Suppose that $x_n \in H$ for a fixed n. Since H is convex x_{n+1} belongs to H.

To study the sequence $\|x_{n+1} - x_n\|$ we start from the following identity:

$$\|x_{n+1} - x_n\| = \|(1-\mu)(x_n - x_{n-1} + \mu(Tx_n\ Tx_{n-1}\|$$

which, using the inner product, becomes

$$\|x_{n+1} - x_n\|^2 = (1 - \mu)^2\|x_n - x_{n-1}\|^2 + 2(1 - \mu)\mu(Tx_n - Tx_{n-1}, x_n - x_{n-1}) + \mu^2\|Tx_n - Tx_{n-1}\|^2.$$

Applying Schwarz's inequalities and taking in account definition (3.2) and (3.3) we have

$$(Tx_n - Tx_{n-1}, x_n - x_{n-1}) \leqslant \|Tx_n - Tx_{n-1}\|\,\|x_n - x_{n-1}\| \leqslant \beta\|x_n - x_{n-1}\|^2.$$

It follows for all $n \geqslant 1$ that

$$\|x_{n+1} - x_n\|^2 \leqslant q(\mu)\|x_n - x_{n-1}\|^2$$

where

$$q(\mu) = (1-\mu)^2 + 2\beta(1-\mu)\mu + \beta^2\mu^2 = (1-\beta)^2\mu^2 - 2(1-\beta)\mu + 1$$

is a quadratic form of variable μ having discriminant $\triangle = 0$. From assumption

$$0 < \mu < \frac{2}{\beta - 1}$$

it follows that

$$0 < q(\mu) < 1.$$

We can prove, by induction, that

$$\|x_{n+1} - x_n\| \leqslant q(\mu)^{(n-1)/2}\|x_1 - x_0\|$$

for all natural number $n \geqslant 1$ and, hence to conclude that $(x_n)_{n\in N}$ is a Cauchy sequence. Since a Hilbert space is a complete metric space, letting

$$x_* = \lim_{n\to\infty} x_n$$

and taking the limit on both sides of

$$x_{n+1} = x_n + \mu(Tx_n - x_n)$$

we obtain

$$x_* = x_* + \mu(Tx_* - x_*).$$

Since $\mu > 0$, therefore we have $Tx_* = x_*$. Thus operator T has at a least a fixed point.

4. Existence and uniqueness for a point boundary value problem.

We are ready to solve problem (1.1) giving a constructive theorem of exitence and uniqueness of the solution under suitable hypotheses. We start with a general uniqueness of the solution.

Theorem 4.1 *If f is a continuous positive and concave function, then the nonlinear two point boundary value problem*

$$(1.1)\quad \begin{cases} u'' + \lambda f(u) = 0 \\ u(0) = u(1) = 0 \end{cases}$$

has at most one positive solution for all $\lambda > 0$.

Proof. Suppose, by contradiction, that there exist two positive solutions u_1, u_2 for some $\lambda > 0$. By Proposition 2.4 these solutions have to be ordered, say $u_1 > u_2$. Remark that function $v = u_1 - u_2$ satisfies the following linear two point boundary value problem

$$(4.11)\quad \begin{cases} v'' + \lambda\psi(t)v = 0 \\ v(0) = v(1) = 0 \end{cases}$$

where

$$\psi = \psi(t) = \frac{f(u_1(t) - f\ u_2(t))}{u_1(t) - u_2(t)}.$$

We have

$$(4.1.2)\quad \begin{cases} -v'' > \lambda\psi(t)v \\ -u''_i = \lambda f(u_i),\ i = 1,2. \end{cases}$$

Multiplying (4.1.2) by u_i and v respectively and integrating any one by parts we get

$$-\int_0^1 v''\, u_i dt > \lambda\psi \int_0^1 v u'_i dt,$$

$$\int_0^1 v'\, u'_i > \lambda \int_0^1 v u_i dt$$

and

$$\int_0^1 u'_i\, v dt = \lambda \int_0^1 f(u_i) dt \quad i = 1,2.$$

From these equalities

$$\lambda \int_0^1 f(u_i) v dt = \int_0^1 u'_i v'\, dt = \lambda\psi \int_0^1 v u_i dt$$

since $\lambda \geqslant 0$ and $v > 0$, it follows that

$$\lambda \int_0^1 (f(u_i) - \psi u_i) v dt = 0$$

and, recalling the definition of ψ, therefore

$$\frac{f(u_i)}{u_i} = \frac{f(u_1) - f(u_2)}{u_1 - u_2}, \quad i = 1,2.$$

On the other hand, since f is concave it must be

$$\frac{f(u_1) - f(u_2)}{u_1 - u_2} \leqslant \frac{f(u_1) - f(0)}{u_1} \leqslant \frac{f(u_2) - f(0)}{u_2}.$$

We deduce that

$$\frac{f(u_i) - f(0)}{u_i} \geqslant \frac{f(u_i)}{u_i}$$

and therefore $f \leqslant (0)$, against hypothesis $f(0) > 0$. We have reached a contradiction: uniqueness of the solution to problem (1.1) is proved.

Theorem 4.2 *Assume that f is continuous function satisfying: there exists $L > 0$ such that*

(i) $f(t) > 0$

(ii) $0 \leqslant \dfrac{f(s) - f(t)}{s - t} \leqslant L$

for all $s, t \geqslant 0$

if $0 < \lambda < \dfrac{\pi^2}{L}$ $\quad \lambda \neq \dfrac{(90)^{1/2}}{L}$

then problem (1.1) has a unique solution in $H^2(0,1)$ given by an iterative sequence.

Proof. Problem (1.1) turns out to be equivalent to the following problem

$$(4.2) \begin{cases} \text{find } u \in L^2(0,1) \\ u(x) = \lambda \int_0^1 G(x,t) f(u(t)) dt \end{cases}$$

where

$$G(x,t) = \begin{cases}(1-x)t, & 0 \leqslant t \leqslant x \\ x(1-t), & x \leqslant t \leqslant 1.\end{cases}$$

We make use of a functional approach to solve problem (4.2). Consider the Hilbert space

$$H = \{u \in L^2(0,1): u(0) = u(1) = 0\}.$$

Let us define the operator $T_\lambda: H \to H$ as

$$T_\lambda u(x) = \lambda \int_0^1 G(x,t) f(u(t))dt.$$

At first, remark that the following two point boundary value problem

$$(1.2) \quad \begin{cases} u'' + \lambda(au + b) = 0 \\ u(0) = u(1) = 0 \end{cases}$$

where $a\, b > 0$, function $f(s) = as + b$ satisfies (i) for $f(0) = b > 0$ and (ii) with $L = a$, has an unique solution if

$$0 \leqslant \lambda < \frac{\pi^2}{L}.$$

We claim that operator T_λ satisfies the hypotheses of Theorem 3.3:

(1) $(T_\lambda u - T_\lambda v, u - v) \geqslant 0$

(2) $\|T_\lambda u - T_\lambda v\| \leqslant \beta\|u - v\|$

where $\beta > 1$ for suitable λ.

Proof of ( ). Setting

$$\phi(t) = \frac{f(u(t)) - f(v(t))}{u(t) - v(t)}.$$

We have

$$\lambda \int_0^1 [u(x) - v(x)]dt \int_0^1 G(x,t)\phi(t)[u(t) - v(t)]dt$$

$$\geqslant \lambda \int_0^1 \int_0^1 G(x,t)\phi[u(t) - v(t)][u(x) - v(x)]dtdx$$

and therefore

$$(T_\lambda u - T_\lambda v, u - v) \geqslant 0.$$

Proof of (2). We have

$$\|Tu(x) - Tv(x)\|^2 \leqslant \lambda^2 \int_0^1 \left(\int_0^1 G(x,t)[f(u(t)) - f(v(t))]dt\right)^2 dx$$

$$\leqslant \lambda^2 L^2 \int_0^1 \left(\int_0^1 G^2(x,t)dt\right)\left(\int_0^1 [u(t)-v(t)]^2 dt\right)dx$$

$$\leqslant \lambda^2 L^2 \|u - v\|^2 \int_0^1 \int_0^1 G^2(x,t)dtdx$$

$$= \lambda^2 L^2 K^2 \|u - v\|^2,$$

where

$$K^2 = \int_0^1 \int_0^1 G^2(x,t)dtdx = 1/90$$

and thus we have

$$\|Tu - Tv\| \leqslant \beta\|u - v\|$$

where $\beta = \lambda LK$.

Using remark on two point boundary value problem (1.2) and Theorem 3.3 we deduce problem (1.1) having a unique solution when

$$\frac{1}{KL} < \lambda \leqslant \frac{\pi^2}{L},$$

given by the following iterative sequence:

$$\begin{cases} u_0(x) \in H \\ u_{n-1} = (1 - \mu)u_n + \mu\lambda \int_0^1 G(x,t) f(u_n(x))dt \end{cases}$$

for

$$0 < \mu < \frac{1}{\lambda LK - 1}.$$

On the other hand if

$$0 < \lambda \leqslant \frac{1}{KL}$$

operator T_λ turns out to be a contraction operator and hence sequence,

$$\begin{cases} u_0(x) \in H \\ u_{n+1}(x) = T_\lambda u_n(x), \end{cases}$$

converges to the unique fixed point.

Corollary 4.3 *Assume f is a twice differentiable function such that for all $t \geqslant 0$*

(i) $f(t) > 0$

(ii) $0 \leqslant f'(t) \leqslant L$

(iii) $f''(t) < 0$

then for $0 < \lambda < \frac{\pi^2}{L}$ *and* $\lambda \neq \frac{(90)^{1/2}}{L}$ *problem* (1.1) *has a unique solution given by an iteration process.*

An example We ask for a solution to the following problem

$$\begin{cases} u'' + \lambda(u + log(1 + u) + 1) = 0 \\ u(0) = u(1) = 0. \end{cases}$$

It is immediate to show that the function $f(t) = t + log(1 + t) + 1$ satisfies the hypotheses of Corollary 4.3 and that nonlinear two point boundary value problem has an unique solution for $0 < \lambda < \frac{\pi^2}{2}$ since one has $f'(t) = 1 + \frac{1}{1+t} \leqslant 2$. Such solutions are given by an iteration process for all

$$0 < \lambda < \frac{\pi^2}{2}, \quad \lambda \neq \left(\frac{45}{2}\right)^{1/2}.$$

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