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(Dedicated to the memory of Professor K. L. Singh)

APPLICATIONS OF FIXED POINT THEOREMS

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Several interesting results using fixed point theory are given in Approximation Theory. An excellent reference is Cheney [5]. For a survey paper one is referred to [9]. The following theorem, due to Brosowski [3], has been a basic important result.

Let f be a contractive linear operator of a normed linear space X. Let C be an f-invariant subset of X and x an f-invariant point. If the set of best C-approximants to x is nonempty, compact and convex, then it contains an f-invariant point. Subrahmanyam [13], S. P. Singh [10], [11] anp K. L. Singh [9] deal with fixed point theorems of nonexpansive mapppings.

Carbone [4] and Rao and Mariadoss [7] deal with generalized nonexpansive mappings Sahab, Khan and Sessa [8] extended the result of [10] in a different direction whereas Hicks and Humphrics relaxed the boundary condition in their paper [6]. The purpose of this note is to give some further results. We give the following definitions.

Let X be a normed linear space and C be a nonempty subset of X. For any $x \in X$, we define

$$d(x,C) = inf \{ \| x-y \| : y \in C\}$$

and an element $y \in C$ is called a best approximation of x by means of elements of C if

$$d(x,C) = \|x-y\|.$$

The set

$$\{y \in C: \|x-y\| = d(x,C)\}$$

is denoted by $P_C(x)$. The set valued mapping $P_C(x)$ is called the *metric projection* associated with C. If $P_C(x)$ is nonempty for every $x \in X$, then C is called *proximinal* The set C is called *Chebyshev set* if the mapping $x \to P_C(x)$ is single-valued; that is, $P_C(x)$ contains at most one element for every $x \in X$.

A function f defined on a convex set C is called *convex* if

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y)$$

$0 \leqslant \lambda \leqslant 1$. If f is convex then the set

$$A + \{x \in C / f(x) \leqslant a, \alpha \in R\}$$

is convex.

Let $x_1, x_2 \in A$. We need to show that

$$\lambda x_1 + (1-\lambda) x_2 \in A.$$

Since f is convex, we get

$$f(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda f(x_1) + (1-\lambda) f x_2) \leqslant \lambda \alpha + (1-\lambda) a = a.$$

Thus

$$\lambda x_1 + (1-\lambda) x_2 \in A$$

i. e., A is convex.

However, the converse is not true.

A convex function need not be continuous.

The first result using convex functions for fixed point theory was given by Belluce and Kiak in 1968 [2].

THEOREM B. *Let C be a nonempty, weakly compact subset of a Banach space X and let T: $C \to C$ be continuous. If*
1. *$(I-T)$ is convex and* 2. *$\inf \|(I-T)x\| = 0$,*
then T has a fixed point.

A functiou f on a convex set C is called *quasi-convex* if

$$f(\lambda x + (1-\lambda)y) \leqslant max(f(x), f(y)).$$

Quasi-convex functions are more general then convex functions. For example, let us consider $C = (0,\infty)$ and $f(x) = \sqrt{x}$.

Then clearly f is quasi-convex, but f is not convex. Taking $x = 1, y = 4$, we get

$$f\left(\frac{x+y}{2}\right) = f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^{\frac{1}{2}},$$

while

$$\frac{1}{2}(f(x) + f(y)) = \frac{1}{2}(1 + 2) = \frac{3}{2} < \left(\frac{5}{2}\right)^{\frac{1}{2}}.$$

A function f is *quasi-convex* on C if and only if

$$A = \{x \in C / f(x) \leqslant \alpha\} \quad \text{is convex.}$$

Proof Let $x_1, x_2 \in C$ and $f(x_2) \leqslant f(x_1)$, $0 \leqslant \lambda \leqslant 1$.
Then by the convexity of A we get

$$f((\lambda x_1 + (1-\lambda)x_1) \leqslant \alpha = f(x_1).$$

This implies that f is quasi-convex on C.

In case f is quasi convex α is any real number, let, $x_1, x_2 \in A$ and let $f(x_2) \leqslant f(x_1)$. Since $x_1, x_2 \in A, f(x_2) \leqslant f(x_1) \leqslant \alpha$. f quasi-convex and C is convex we have, for $0 \leqslant \lambda \leqslant 1$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leqslant f(x_1) \leqslant \alpha$$

Hence, $(1-\lambda)x + \lambda x_1 \in A$, i.e., A is convex.

In case $f(\lambda x_1 + (1-\lambda)x_2) \leqslant max\,(f(x_1), f(x_2))$,
the f is also called *strongly quasi-convex function.*

The condition

$$f\left(\frac{x+y}{2}\right) \leqslant max\,(f(x), f(y))$$

always implies uniqueness; i.e., a strongly quasi-convex function can achieve its minimum over a convex set C at no more than one point.

In fact, if

$$f(x_0) = f(x_0) = \inf_{x \in C} f(x),$$

then

$$f\left(\frac{x_0 + y_0}{2}\right) \leqslant max\ (f(x_0), f(y_0));$$

contradiction

The following result is due to Singh and Veitch [12].

Theorem S - *Let C be a nonempty, weakly compoct, convex subset of a Banach space X If T: C→C is continuous, (I-T) is quasi convex on C and*

$$in\ f_{x \in C} \|(I-T)x\| = 0,$$

then T has at least one fixed paint.

We incldue the proof for the sake of completeness.

Proof If a quasi-convex functional defined on a weakly closed convex set C is lower semicontinuous, then it is weakly lower semicontinuous.

Consider

$$J(x) = \|(I-T)x\|.$$

I-T quasi-convex implies that J is quasi convex I-T lower semicontinuous hence J is weakly lower semicontinuous. Since C is weakly compact J attains its minimum, i.e., there is an $x_0 \in C$ such that

$$\|(I-T)x_0 = \inf_{x \in C} \|(I-T)x\|.$$

But $\inf_{x \in C} \|(I-T)x\| = 0$, therefore $\|(I-T)x_0\| = 0$.

So $$Tx_0 = x_0.$$

Corollary 1. *Let C be a nonempty weakly compact convex subset of a Banch space X. If : C→C is nonexpansive map and (I-T) is quasi-convex, then f has a fixed point.*

Remark. Since a nonexpansive map is continuous and

$$inf\ \{\|(I-f)x\| : x \in C\} = 0,$$

so the result follows from Theorem S.

Theorem 1. *Let C be a subset of a Banach space X and $T: X \to X$ a map such x_0 is invariant under T and $T(\partial C) \subseteq C$. If the set D of best C-approximants to x_0 is weakly compact, convex, T is nonexpansive on $D \cup \{x_0\}$ and $(I-T)$ is quasiconvex, then T has a fixed point closest to x_0.*

Proof. Fist of all we prove that $T: D \to D$

In fact, if $y \in D$, then y is closest to x_0.

Now

$$\|Ty-x_0\| = \|Ty-Tx_0\| \leqslant \|y-x_0\|.$$

This implies that $Ty \in D$.

Since D is nonempty, weakly compact, convex, $T: D \to D$ is nonexpansive and $(I\ T)$ is quasiconvex so T has a fixed point in D closest to x_0.

Note. In Theorem 1 we do not require that T is nonexpansive and linear as in [4], and $T: C \to C$ is not needed.

Theorem 2. *Let X be a Banach space and $T: X \to X$ a map such that* $Tx_0 = x_0$ and $T(\partial C) \subseteq C$.

Further, if the set D of best C-approximants to x_0 is closed bounded and convex, T is a noncxpansive map on $D \cup \{x_0\}$ and $(I\ T)(D)$ is closed then T has a fixed point closest to x_0

For our proof, we need the following

Theorem : *Let X be Banach space and C a closed bounded convex subset of X. If $f: C \to C$ is nonexpansive and $(I-f)C$ is a closed of X, then f has a fixed point in C.*

Proof. It is easy to show as in Theorem 1 that $T: D \to D$.

T has a fixed point in D by the above theorem, say $Tu = u$, $u \in D$. Clearly, u is closest to x_0.

Note. In the above Theorem 2, T need not be even continuous on X. T should have desired conditions to garantee a fixed point only on D.

We need the following in our Theorem 3.

Theorem B ([1]) *Let $C \neq \Phi$ weakly compact convex subset of a Banach space X and $f: C \rightarrow C$ a nonexpansive mapping. If there is a strictly increasing function*

$$g: [0,\infty) \rightarrow [0,\infty)$$

with $g(0) = 0$, such that for all $x, y \in C$ and $0 \leqslant \lambda \leqslant 1$, we have

$$\| f(\lambda x + (1-\lambda)y) - \lambda x - (1-\lambda)y \| \leqslant g(\|x - fx\| + \|y - fy\|),$$

which is weaker than convexity condition.

Then f has a fixed point.

Theorem 3. *Let X be a Banach space and $T: X \rightarrow X$ a map. Let $Tx_0 = x_0$ and $T(\partial C) \subset C$. Assume that the set D of best C-approximants to x_0 is nonempty, weakly compact, convex and T is nonexpansive on $D \cup \{x_0\}$. Furthermore, if there is a strictly increasing function $g:[0,\infty) \rightarrow [0,\infty)$ with $g(0) = 0$ such that for all $x, y \in C$ and $0<\lambda<1$, we have*

$$\|T(\lambda x + (1-\lambda)y) - \lambda x - (1-\lambda)y\| \leqslant g(\|x-Tx\| + \|y-Ty\|),$$

then T has a fixed point closest to x_0.

Proof. It is easy to show that $T: D \rightarrow D$, and then using the above Theorem B we get that T has a fixed point in D, closest to x_0.

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