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(Dedicated to the memory of Professor K. L. Singh)

A FIXED POINT THEOREM IN UNIFORMLY CONVEX SPACES

By

Anil Rajput

Department of Mathematics, S. V. College, Bairagarh (Bhopal) 462030, India

And

M. S. Rathore

Govt Postgraduate College, Sehore, M. P.

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Bogin [1] and Rhoades [8] have generalized a fixed point theorem in uniformly convex spaces of Goebel, Kirk and Shimi [6]. In this paper, to present another extension of their result, we establaih the following theorem;

THEOREM. Let X be a uniformly convex Banach space, K a non-empty bounded closed and convex subset of X, and $F:K\to K$ a continuous mapping satisfying for $x,y \in K$:

(1) $$\|F(x)-(y)\| \leq a\|x-y\| + \beta(\|x-F(x) + \|y-F(y)\|) + \gamma(\|x-F(y)\| + |y-F(x)\|) + \sigma(\|F.y) - F^2(x)\| + \|x-F^2(x)\| + \eta(\|F(x) - F^2(x)\| + \|y-F^2(x)\|)$$

where $\alpha, \beta, \gamma, \sigma$ and η are non-negative real numbers such that;

(2) $\alpha+2(\beta+\gamma+\sigma+\eta)=1,\ \alpha+\beta+2\gamma+\sigma+\eta \leq 1$ and $\alpha+2\beta+2\eta \leq 1.$

Then F has a fixed point in K.

It may be mentioned that the result in [6] is obtained by taking $\gamma=\sigma=0$ in (1). We shall follow the same line of argument and notation as that of [6].

Proof. Putting $y=F(x)$ in (1), we get

$$\| F(x) - F^2(x) \| \leqslant \alpha \| x - F(x) \| + \beta (\| x-F(x) + \| F(x) - F^2(x) \|)$$

$$+ \gamma (\|x-F^2(x)\| + \| F(x) - F(x)\|)$$

$$+ \sigma (\|F^2(x) - F^2(x) \| + \| x-F^2(x) \|)$$

$$+ \eta (\|F(x)-F^2(x) \| + F(x)-F^2(x) \|)$$

$$\leqslant (\alpha+\beta) \| x-F(x) \| + (\beta+2\eta) \| F(x-F^2(x) \|$$

$$+ (\gamma+\sigma) \| x-F^2(x) \|$$

$$\leqslant (\alpha+\beta) \| x-F(x) \| + (\beta+2\eta) \| F(x)-F^2(x) \|$$

$$+ (\gamma+\sigma) (\| x-F(x) \| + F(x)-F^2(x) \|)$$

$$\| F(x) - F^2(x) \| \leqslant \frac{\alpha+\beta+\gamma+\sigma}{1-\beta-\gamma-\sigma-2\eta} \| x-F(x) \|.$$

By (2), $\frac{\alpha+\beta+\gamma+\sigma}{1-\beta-\gamma-\sigma-2\eta} = 1$. Therefore

(3) $\| F(x)-F^2(x) \| \leqslant \| x-F(x) \|$,

and consequently

(4) $\| F^{i+1}(x)-F^i(x) \| \leqslant F^i(x)-F^{i-1}(x) \| \quad i=1,2,3...$

Assertion: $\inf_{x \in K} \| x-F(x) \| = 0.$

To prove the assertion, we assume $\inf_{x \in K} \| x-F(x) \| = d > 0.$

Let $\epsilon > 0$ and choose $x \in K$ such that $\| x-F(x) \| \leqslant d+\epsilon$ using uniform convexity and asumptions on K, we can find a real number p, $0<p<1$, as in [6] such that

(5) $\| F^{i-1}(x)-F^{i+1}(x) \| \leqslant 2p \| F^{i-1}(x)-F^i(x) \|, \ 0<p<1$

Writing j for $i-1$ in (5), we obtain

(6) $\| F^j(x)-F^{j+2}(x) \| \leqslant 2p \| F^j(x)-F^{j+1}(x) \|, \ 0<p\leqslant 1$

Case 1. γ or $\sigma \neq 0$.

By (1)

$$\|F^i(x)-F^{i+1}(x)\| \leqslant (\alpha+\beta)\|F^i(x)-F^{i-1}(x)\| + (\beta+2\eta)\|F^i(x)-F^{i+1}(x)\|$$
$$+ (\gamma+\sigma)\|F^{i+1}(x)-F^{i-1}(x)\|$$

Therefore by (5),

$$(1-\beta-2\eta)\|F^i(x)-F^{i+1}(x)\| \leqslant (\alpha+\beta+2p(\gamma+\sigma))\|F^i(x)-F^{i-1}(x)\|$$

hence $(1-\beta-2\eta)\, d \leqslant (\alpha+\beta+2p(\gamma+\sigma))(d+\epsilon)$.

Since $\epsilon > 0$ is arbitrary, we have

$(1-\beta-2\eta)\, d \leqslant (\alpha+\beta+2p(\gamma+\sigma))\, d.$

This implies

$(1-\beta-2\eta \leqslant \alpha+\beta+2p(\gamma+\sigma)$

which, in view of $p<1$ contradicts (2).

Case II. $\gamma=\sigma=0$ Let

$m=[F^i(x)+F^{i+1}(x)]/2$. Then by (1),

$$\|m-F(m)\| \leqslant 2^{-1}\|F^i(x)-F(m)\| + 2^{-1}\|F^{i+1}(x)-F(m)\|$$
$$\leqslant 2^{-1}[\alpha\|F^{i-1}(x)-m\|+\beta(\|F^{i-1}(x)-F^i(x)\|+\|m-F(m)\|]$$
$$+\eta[\|F^i(x)-F^{i+1}(x)\|+\|m-F^{i+1}(x)\|)]$$
$$+2^{-1}[\alpha\|F^i(x)-m\|+\beta(\|F^i(x)-F^{i+1}(x)\|+\|m-F(m)\|]$$
$$+\eta[\|F^{i+1}(x)-F^{i+2}(x)\|+\|m-F^{i+2}(x)\|)].$$

Therefore

$$(1-\beta)\|m-F(m)\| \leqslant 2^{-1}[\alpha(\|F^{i-1}(x)-m\|+\|F^i(x)-m\|)$$
$$+\beta(\|F^{i-1}(x)-F^i(x)\|+\|F^i(x)-F^{i+1}(x)\|)$$
$$+\eta\,\|F^i(x)-F^{i+1}(m)\|+\|m-F^{i+1}(x)\|+\|F^{i+1}(x)-F^{i+2}(x)\|$$
$$+\|m-F^{i+2}(x)\|)]$$

$$\leqslant 4^{-1}(\|F^{i-1}(x)-F^{i}(x)\| + \|F^{i-1}(x)-F^{i+1}(x)\|$$

$$+\|F^{i}(x)-F^{i+1}(x)\|) + 2^{-1}[\beta\ F^{i-1}(x)-F^{i}(x)\|$$

$$+\|F^{i}(x)-F^{i+1}(x)\|) + \eta'\|F^{i}(x)-F^{i+1}(x)\|$$

$$+\|F^{i+1}(x)-F^{i+2}(x)\| + 2^{-1}(\|F^{i}(x)-F^{i+1}(x)\|$$

$$+\|F^{i+1}(x)-F^{i+2}(x)\| + \|F^{i}(x)-F^{i+2}(x)\|)],$$

Hence applying (5) and (6) we have

$$(1-\beta)d < \tfrac{1}{4}(\alpha+2\alpha p+\alpha+\eta+\eta+2\eta p)\ (d+\varepsilon) + \tfrac{1}{2}(\beta+\beta+\eta+\eta)\ (d+\varepsilon)$$

letting $\epsilon \to 0$ we obtain

$$(1-\beta)d \leqslant \tfrac{1}{4}(2\alpha\,(1+p) + 2\eta(1+p))\, d + (\beta+n)d$$

that is, if α or $\eta \neq 0$,

$$(1-\beta) < \alpha+\beta+2\eta,$$

which contradicts (2).

Case III. $\alpha = \gamma = \sigma = \eta = 0$

In view of Soardi's result [9] this case need not be considered. Thus the assertion is proved.

Now for $\varepsilon \epsilon (0,1)$ assume

$$C_\varepsilon = \{x: \|x-Fx\| \leqslant \varepsilon\} \text{ and}$$

$$D_\varepsilon = \{x \in C_\varepsilon: \|x\| \leqslant \bar{a} + \varepsilon\},$$

where $\bar{a} = \lim_{\varepsilon \to 0} a(C_\varepsilon)$ and $a(C_\varepsilon) = \inf\{\|x\|: x \in C_\varepsilon\}$

Since F is continuous the sets C_ε and (S_0) D_ε are closed and by the assertion they are non-void. The proof is completed by showing

$$\bigcap_{\varepsilon>0} C_\varepsilon \neq \phi.$$

If $\bar{a} = 0$ then $0 \in \bigcap_{\varepsilon>0} C_\varepsilon \neq \phi$. Therefore we may assume $\bar{a} > 0$,

letting $u_1, u_2 \in C_\varepsilon$ we have by (1) and the triangle inequality for $i = 1,2$.

$$(7)\quad \|u_1-F((u_1+u_2)/2\| \leqslant \|u_1-F(u_1)\| + \|F(u_1)-F(u_1+u_2)/2)\|$$

$$\leqslant \epsilon + \alpha\|u_1-(u_1+u_2)/2\| + \beta(\|u_1-F(u_1)\|$$

$$+\|(u_1+u_2)/2-F((u_1+u_2)/2\|) + \gamma(\|u_1-F((u_1+u_2)/2\|$$

$$+\|(u_1+u_2)/2-u_1\| + \|u_1-F(u_1)\|) + \sigma(\|u_1-F(u_1)\|$$

$$+\|F((u_1+u_2)/2)-u_1\| + \|u_1-F(u_1)\| + \|F(u_1)-F^2(u_1)\|)$$

$$+\eta(\|F(u_1)-F^2(u_1)\| + \|(u_1+u_2)/2-u_1\|$$

$$+\|u_1-F(u_1)\| + \|F(u_1)-F_2(u_1)\|)$$

We note by (3) that

$\|F(u_1)-F^2(u_1)\| \leqslant \|u_1-F(u_1)\| \leqslant \varepsilon.$

Also since

$\|(u_1+u_2)/2-F((u_1+u_2)/2)\| \leqslant \max_{i=1,2} \|u_i-F((u_1+u_2)/2)\|.$

We obtain

$$(1-\beta-\gamma-\sigma) \max_{i=1,2} \|u_i-F((u_1+u_2)/2)\|$$

$$\leqslant (1+\beta+\gamma+3\sigma+3\eta)\,\varepsilon + (\alpha+\gamma+\eta)\,\|u_1-u_2\|/2$$

So $\|u_1-F((u_1+u_2)/2\| \leqslant q\varepsilon + r\|u_1-u_2\|/2,$

where $q = (1+\beta+\gamma+3\sigma+3\eta)/(1-\beta-\gamma-\sigma)$ and $r = (\alpha+\gamma+\eta)/(1-\beta-\gamma-\sigma).$

By (2), $r < 1$, therefore

$$\|u_1-F((u_1+u_2)/2\| < q\varepsilon + \|u_1-u_2\|/2.$$

Now the rest part of proof follows as in [6] we have remarked that if β or σ or $\eta \neq 0$ in (2) then the fixed point is unique.

REMARK: If X is compact metric space and F a continuous mapping of X into itself, satisfying the inequality,

$$d(Fx,Fy) < \alpha d(x,y) + \beta\{d(x,Fx)+d(y,Fy)\} + \gamma\{d(x,Fx)+d(y,Fx)\}$$
$$+\sigma\{d(Fy,F^2x)+d(x,F^2x)\} + \eta\{d(Fx,F^2x)+d(y,F^2x)\},$$

where α, β, γ, σ and η are non-negative real numbers such that,

$\alpha+2(\beta+\gamma+\sigma+\eta) \leqslant 1$ and $\gamma+\sigma+\eta > 0$ Then F has a unique fixed point.

We further obtain from this remark:

(i) A result of Khan and Sharma [7] if we put $\eta=0$.

(ii) A result of Edelstein [2] if we put $\beta=\gamma=\sigma=\eta=0$ and $\alpha=1$.

(iii) A result of Fisher [3] if we put $\alpha=\gamma=\sigma=\eta=0$ and $\beta=\frac{1}{2}$.

(iv) A result of Fisher [4] if we put $\alpha=\beta=\sigma=\eta=0$ and $\gamma=\frac{1}{2}$.

(v) A result of Fisher [5] if we put $\alpha=a$, $\beta=b$, $\gamma=c$ and $\eta=\sigma=0$.

REFERENCES

[1] J. Bogin, A generalization of a fixed point theorem of Goebel, Kirk and Shimi, *Canad. Math. Bull.* **19** (1976), 7-12.

[2] M. Edelstein, On fixed and periodic points under contractive mappings, *J. Lond. Math. Soc.* **37** (1962), 74-79.

[3] B. Fisher, A fixed point mapping, *Bull. Calcutta, Math. Soc.* (1976).

[4] B. Fisher, A fixed point theorem for compact metric spaces, *Publ. Math.* (1976).

[5] B. Fisher, On three fixed point mappings for compact metric spaces, *Indian J. Pure Appl Math.* **8** No. 4 (1977), 479-481.

[6] K. Goebel, W. A. Kirk and T. N. Shimi, A fixed point theorem in uniformly convex spaces, *Boll. U. M. I.* (4), **7** (1976), 67-75.

[7] S. Khan, and P. L. Sharma, Fixed point mappings for compact metric space, *The Mathematics Education* **24** 1 (1990).

[8] B. E. Rhoades, Some fixed point theorems in Banach spaces, *Comment. Math. Univ. St. Pauli* **24** -1 (1975), 13-16.

[9] R. Soardi, Su un problem di punto unito di S. Reich. *Boll, Un. Mat. Ital.* (4), **44** (1971), 841-845.