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VISCO-ELASTIC BOUNDARY LAYER FLOW PAST A STRETCHING PLATE WITH SUCTION AND HEAT TRANSFER

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ABSTRACT

In this paper, we have studied the Nusselt number for the convective heat transfer for leminar boundary layer flow of 'Walters liquid B' past a stretching plate with constant suction and with uniform temperature T. Further, we have obtained the temperature profile in the case of plate temperature $T=T_{\infty}+Ax^{n}$. In each case, the Nusselt number has been derived for Prandtl number $p<<1$, $p=1$ and $p>>1$.

1. INTRODUCTION

In day to-day problems, we come across the problems of flow past a non-rigid plate or boundary where suction is also present. The study of various aspects of suction is very useful in the design of aircraft wings. In more recent times one of the applications of suction is to reduse drag. In 1904, Prandtl considered suction in boundary layer control to-prevent separation. Siddappa and Khapate [6] extended the study Crane [2] to the visco-elastic flow past a stretching plate. In 1980, Siddappa and Hiremath [7] studied the problem of Siddappa and Khapate [6] with suction. Siddappa and Abel [5] worked on the boundary layer flow of Walters liquid B past a stretching plate with suction and heat transfer.

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In the present paper, the problem of Siddappa and Abel [5] is extended to study the Nusselt number for convective heat transfer.

2. Formation of the Problem

Let X-axis be along the plate in the direction of fluid motion, Y-axis be perpendicular to the plate and the slit be at origin. The boundary layer equation governing the flow of visco-elastic fluid (Walters liquid B) ([1], [4]) is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} - K_0^* \left\{\frac{\partial^3 u}{\partial t \partial y^2} + u\frac{\partial^3 u}{\partial x \partial y^2} + v\frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}\right\} \tag{2.1}$$

In the case of steady state boundary layer flow past a stretching plate with suction for the short memory approximation of Walters liquid B, the equation of motion and equation of continuity are [5];

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial^2 y} - K_0^* \left\{ u\frac{\partial^3 u}{\partial x \partial y^2} + v\frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}\right\} \tag{2.2}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.3}$$

respectively .

Boundary conditions are:

$$y = 0\,;\ u = mx,\ v = -v_0$$
$$y \to \infty\,;\ u = 0,\ v = -C,\ C > 0\,,\ u_y = 0 \tag{2.4}$$

3. Solution of the Problem

Introducing dimensionless numbers

$$y' = \frac{y}{h}, u' = \frac{uh}{\nu}, x' = \frac{x}{h}$$

$$v' = \frac{vh}{\nu} \quad K_0^{*\prime} = \frac{k_0^*}{h^2}$$

and substituting all these in equations (2.2) — (2 4), we have

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} - K_0^* \left\{ u\frac{\partial^3 u}{\partial x \partial y^2} + v\frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} \right\} \tag{3.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.2}$$

and boundary conditions:

$$y = 0, \; u = mx, \; v = -v_0$$
$$y \to \infty, \; u = 0, \; v = -C, \; C > 0, \; u_y = 0 \tag{3.3}$$

where dash has been suppressed for the convenience.

Setting the similarity solution of the form

$$u = mxf'(y), \; m > 0 \tag{3.4}$$

we get

$$v = -\{mf(y) + v_0\} \tag{3.5}$$

Putting (3.4) and (3.5) in (3.1), we get

$$f'^2(y) = f''(y) f(y) - \frac{v_0}{m} f''(y) = \frac{1}{m} f'''(y) - K_0^*$$
$$\{2f'(y) f'''(y) - f(y) f^{iv}(y) - \frac{v_0}{m} f^{iv}(y) - f''^2(y)\} \tag{3.6}$$

and boundary conditions:

$$y = 0; \; f = f' = 1$$
$$y \to \infty; \; C = mf + v_0, \; f'' = 0 \tag{3.7}$$

Trying the solution assuming

$f'(y) = e^{-ry}, r>0$

we get r as the root of cubic equation

$$\frac{v_0}{m} K_0^* r^3 - (\frac{1}{m} - K_0^*)r^2 + \frac{v_0}{m} r + 1 = 0 \tag{3.8}$$

Hence the velocity components are

$$u = mx\, e^{-ry} \tag{3.9}$$

and

$$v = -\frac{m}{r}(1-e^{-ry})-v_0 \tag{3.10}$$

4. Heat Conduction in Linear Stretching Plate

The boundary layer energy equation is

$$\rho c_p(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = K\frac{\partial^2 T}{\partial y^2} \tag{4.1}$$

and boundary conditions:

$$y = 0;\ T = T_p$$

$$y \to \infty;\ T = T_s \tag{4.2}$$

where T_p is plate temperature and T_s is surrounding temperature.

Introducing dimensionless variables

$$\eta = ry \text{ and } \theta = \frac{T-T_s}{T_p-T_s} \tag{4.3}$$

the equation (4.1) becomes

$$\left[-\frac{m}{r}(1-e^{-ry})-v_0\right]\frac{\partial\theta}{\partial\eta} = \frac{r}{p}\frac{\partial^2\theta}{\partial\eta^2} \tag{4.4}$$

and boundary conditions reduce to

$$\eta = 0;\ \theta = 1$$

$$\eta \to \infty;\ \theta = 0 \tag{4.5}$$

The solution of equation (4.4) under the boundary condition (4.5) is

$$\theta(\eta) = \frac{\int_{\eta}^{\infty} exp\left\{-\frac{m}{r^2}P(\eta + e^{-\eta} + \frac{v_0 r}{m}\eta)\right\} d\eta}{\int_{\eta}^{\infty} exp\left\{-\frac{m}{r^2}P(\eta + e^{-\eta} + \frac{v_0 r}{m})\right\} d\eta} \tag{4.6}$$

Putting $\frac{m}{r^2} Pe^{-\eta} = t$, we have

$$\theta(\eta) = \frac{\gamma(\frac{p}{r}(m+v_0 r), \frac{m}{r^2}Pe^{-\eta})}{\gamma(\frac{P}{r^2}(m+v_0 r), \frac{m}{r^2}P)} \tag{4.7}$$

For $P = 1$, $\theta(\eta)$ becomes

$$\theta(\eta) = \frac{\gamma(\frac{1}{r^2}(m+v_0 r), \frac{m}{r^2}e^{-\eta})}{\gamma(\frac{1}{r^2}(m+v_0 r), \frac{m}{r^2})} \tag{4.8}$$

Differentiating (4.7) with respect to η, we get

$$\left.\frac{\partial\theta}{\partial\eta}\right|_0 = \frac{-(\frac{mp}{r^2})^{1/r^2(m+v_0 r)p} exp(-\frac{m}{r^2}P)}{\gamma(\frac{P}{r^2}(m+v_0 r), \frac{m}{r^2}P)} \tag{4.9}$$

The Nusselt Number

The Nusselt number $Nu(x)$ is defined by

$$Nu(x) = -(\frac{\partial T}{\partial y})_{y=0} / T_p - T_s$$

$$= -(\frac{\partial\theta}{\partial\eta}).(\frac{\partial\eta}{\partial y})_0 \tag{4.10}$$

Hence, for $p=1$, the Nusselt number

$$Nu(x) = \frac{\left(\frac{m}{r^2}\right)^{1/r^2(m+v_0r)} exp\left(-\frac{m}{r^2}\right)}{\gamma\left(\frac{1}{r^2}(m+v_0r), \frac{m}{r^2}\right)} \tag{4.11}$$

For $P \ll 1$, using series expansion for the incomplete Gamma function,

$$\gamma(x,a) \doteq a^x \sum_{n=0}^{\infty} \frac{(-1)^n a^n}{(x+n)n!},$$

we have

$$Nu(x) \doteq \frac{m}{r} p \quad \text{for small } p \tag{4.12}$$

For $P \gg 1$, applying stirling approximation formula for large p, we obtain

$$Nu(x) = \frac{\left(\frac{m}{r^2}p\right)^{p/r^2(m+v_0r)} exp\left(\frac{p}{r} v_0\right)}{(2\pi)^{1/2}\left(\frac{p}{r^2}(m+v_0r)\right)^{p/r^2(m+v_0r)-\frac{1}{2}}} \tag{4.13}$$

5. Heat Conduction for Linear Stretching Plate with Temperature $T_p = T_s + Ax^n$

Recalling the similarity variables $\eta = ry$ and defining the dimensionless temperature by $\frac{T-T_s}{T_s} = Ax^n g(\eta)$ and substituting in the boundary layer equation of temperature (4.1), we have

$$\frac{m}{r^2} p \left[\eta e^{-\eta} g(\eta) - \left(1-e^{-\eta}+\frac{v_0r}{m}\right) g'(\eta)\right] = g''(\eta) \tag{5.1}$$

and boundary condition:

$$\eta = 0;\ g = 1$$
$$\eta = \infty;\ g = 0 \tag{5.2}$$

Special case:

The solution of the equation (5.11) for $p = 1$, $\eta = 0$ subject to the boundary conditions (5.2) is

$$g(\eta) = \frac{\gamma'(\frac{1}{r^2}(m+v_0 r), \frac{m}{r^2} e^{-\eta})}{\gamma(\frac{1}{r^2}(m+v_0 r), \frac{m}{r^2})} \tag{5.3}$$

To solve (5.1) for general n, we put $e^{-\eta} = t$, then the equation (5.1) reduces to

$$t\frac{d^2g}{dt^2} + (1 + \frac{m}{r^2}(t - \frac{v_0 r}{m} - 1))\frac{dg}{dt} - \frac{m}{r^2} png = 0 \tag{5.4}$$

We find the series solution of (5.4)

$$g(t) = \sum_{k=0}^{\infty} a_k t^{k+s}, \; a_0 \neq 0 \tag{5.5}$$

together with boundary conditions:

$$g = 1 \; at \; t = 1$$
$$g = 0 \; at \; t = 0 \tag{5.6}$$

given by

$$g(t) = a_1 t^{b_1 p} F(b_1 p - n; 1 + b_1 p, \frac{-m}{r^2} pt) \tag{5.7}$$

where

$$a_1 = \frac{1}{F(b_1 p - n, 1 + b_1 p, \frac{-m}{r^2} p)}$$

$$b_1 = \frac{m}{r^2}(1 + \frac{v_0 r}{m})$$

and $F(a,b,x) = 1 + \frac{a}{b} x + \frac{a(+1)}{b(b+1)} \frac{x^2}{2} + \ldots$, the confluent hypergeometric series.

Special Case: When $p = \frac{n}{b_1}$, we have

$$F(b_1 p-n,\ 1+b_1 n,\ \frac{-m}{r^2} pt) = 1 \text{ for all } t, \text{ and } a_1 = 1.$$

Then,

$$g(\eta) = e^{-b_1 p\eta},\ g(0) = 1,\ g'(\eta) = -b_1 p\, e^{-b_1 p\eta}, g'(0) = -b_1 p,$$

The Nusselt number is

$$Nu(x) = -\left(\frac{g'(\eta)}{g(\eta)}\right)_{\eta=0} = 1/r(m+v_0 r)p \tag{5.8}$$

6. Temperature Profile for Small p .

From the equation (5.7), we have

$$g(\eta) = a_1 e^{-b_1 pn}\Big[1 + \frac{m}{r^2} np\ e^{-\eta} + p^2\Big\{\frac{m}{4r^2}\ e^{-2\eta}(n^2-n)$$

$$-\frac{m^2}{r^2}(1+\frac{v_0 r}{m})e^{-\eta}(n+1)\Big\} + o(p^3)\Big].$$

Differentiating with respect to η and putting $\eta = 0$, we get

$$g'(0) = -P\Big[b_1 + \frac{m}{r^2} n + p\Big\{-\frac{m^2}{r^4}\Big(\frac{n^2+n}{2} + (1+\frac{v_0 r}{m})n + \frac{v_0 r}{m} + 1\Big)\Big\}$$

$$+ o(p^2)\Big] \tag{6.2}$$

The Nusselt number is

$$Nu(x) = -\left(\frac{g'(\eta)}{g(\eta)}\right)_{\eta=0} \cdot r$$

$$= \frac{p}{r}\ [m(n+1) + p\ \Big\{\frac{-m^2}{2}\Big(\frac{n^2+n}{2} + (1+\frac{v_0 r}{m})n + \frac{v_0 r}{m} + 1\Big)\Big\}$$

$$+ o(P^2)] \tag{6.3}$$

7. Conclusion and Discussion.

Taking the values of $v_0 = 0$ and $K_0^* = 0$ in the equation (3.8), we get $r = \sqrt{m}$ which is identical to $\left(\frac{u_p}{\nu x}\right)^{1/2}$ where $u_p = ax$ in [2], in dimensional form and then the expressions (3.9), (3.10) and (4.6) reduce to the expressions of velocty components and temperature distribution obtained by Carragher, [3]. Also, we get:

(a) for $p = 1$; $Nu(x) = \left(\frac{u_p}{\nu x}\right)^{1/2} (e-1)^{-1}$

(b) for $p << 1$; $Nu(x) = \left(\frac{u_p}{\nu x}\right)^{1/2} p$ and

(c) for $p >> 1$; $Nu(x) = (u_p\, p/\nu x)^{1/2}$,

which are the results (4.21), (4.25) and (4.28) respectively of Carragher, [3].

(2) From (5 3), (5.8) and (6.3)

(a) for $p = 1$, $n = 0$;

$$Nu(x) = \left(\frac{u_p}{\nu x}\right)^{1/2} (e-1)^{-1}$$

(b) for $p = n$;

$$Nu(x) = p\left(\frac{u_p}{\nu x}\right)^{1/2}, \text{ and}$$

(c) for $p << 1$;

$$Nu(x) = p\left(\frac{u_p}{\nu x}\right)^{1/2} [(n+1) + p\,(-\frac{n^2}{2} - \frac{3n}{2} - 1) + o\,(p^2)]$$

which are the results (4.85), (4.86) and (4.87), respectively, of Carragher [3].

When the plate is rigid then $v_0 = 0$. The root of the equation (3.8) is $r = \dfrac{1}{(1/m)^{1/2} - K_0^*}$ and the results (4.11), (4.12) and (4.13) become

(a) for $p = 1$, $Nu(x) = \dfrac{\left(\dfrac{m^2}{1-K_0^* m}\right)^{m^2/(1-K_0^* m)} exp\{-m^2/(1-K_0^* m)\}}{\gamma\left(\dfrac{m^2}{1-K_0^* m}, \dfrac{m^2}{1-K_0^*}\right)} \left(\dfrac{m}{1-K_0^* m}\right)^{1/2}$

(b) for $p << 1$, $Nu(x) \doteq \left(\dfrac{m^2}{1/m - K_0^*}\right)^{1/2} p$, and

(c) $p >> 1$, $Nu(x) = m\left(\dfrac{mp}{2(1-K_0^* m}\right)^{1/2}$,

where $Nu(x)$ is well defined for $K_0^* m < 1$ and dependency of $Nu(x)$ on visco-elasticity K_0^* is clarified. In case with plate temperature $T = T_s + Ax^n$, the results (5.3), (5.8) and (6 3) for the Nusselt number reduce to:

(a) for $p = 1$, $n = 0$;

$$Nu(x) = \frac{[m^2/(1-K_0^*)]^{(m^2/1-K_0^*)} \cdot exp\{-\dfrac{m^2}{(1-K_0 m)}\}}{\gamma\left(m/(1-K_0^* m), m/(1-K_0^* m)\right)} \left[m/(1-K_0^* m)\right]^{1/2}$$

(b) for $p = n/b_1$;

$$Nu(x) = n\left(\frac{m}{1-K_0^* m}\right)^{1/2}, \text{ and}$$

(c) $Nu(x) = p\left(\frac{m}{1-K_0^* m}\right)^{1/2}[m(n+1) - p(m^2/r^2(n^2+n/2+n+1))$

$+ o(p^2)]$

$$n = 0,\ Nu(x) = p\left(\frac{m}{1-K_0^* m}\right)^{1/2}\left[m - \frac{m^3}{1-K_0^* m}p + o(p^2)\right]$$

$$n = 1,\ Nu(x) = p\left(\frac{m}{1-K_0^* m}\right)^{1/2}\left[2m + \frac{m^3}{1-K_0^* m}p(-3) + o(p^2)\right]$$

$$n = 2,\ Nu(x) = p\left(\frac{m}{1-K_0^* m}\right)^{1/2}\left[3m - \frac{8m^3}{1-K_0^* m}p + o\ p^2)\right]$$

where the coefficient of convective heat transfer depends on n as well visco elasticity K_0^*.

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REFERENCES

[1] K. Walters, Developments in non-Newtonian fluid mechanics, *J. Non-Newtonian Mech.* 5 (1979), 113–124.

[2] L. J. Crane, Flow past stretching plate, *ZAMP* 21 (1970), 645-647.

[3] P. Carragher, Ph.D. thesis, University of Dublin, 1978, 41-56.

[4] S. K. Pathak and M. C. Chaturvedi, Leminar boundary layer over a flexible surface in oscillatory, flow, Proc. Indian Acad. Sci. Sect. A 8 (1975), 179–196.

[5] B. Siddappa and M. S. Abel, Visco-elastic boundary layer flow past a stetching plate with suction and heat transfer, *Rheol Acta.* **25** (1986), 319–320.

[6] B. Siddappa and B. S. Khapate, Rivlin-Ericksen fluid flow past a stretching plate, *Rev. Roumaine Sci. Tech. Mech. Appl.* **1** (1976), 497-505.

[7] B. Siddappa and P. S. Hiremath, Rivlin-Ericksen fluid flow past a stretching plate, with suction, *Kyungpook Math. J.* **20** (1980), 267.