Jñānābha, Vol, 21, 1991

CAUCHY'S PROBLEM FOR PARTIAL DIFERENTIAL EQUATIONS WITH BESSEL-TYPE DIFFERENTIAL OPERATORS

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(*Received: May 31, 1991; Revised: September 18, 1991*)

ABSTRACT

We obtain existence theorems for the solutions of certain linear parabolic equations which contain different Bessel-type differential operators. These operators are infinitesimal generators of semigrops, these being differentiable and analytic. The semigroups are characterized in the early paragraphs. The equations analysed appear as particular cases of the Fokker-Planck equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left[A(x)\frac{\partial}{\partial x}[u(x,t)]\right]+\frac{\partial}{\partial x}\left[B(x)\,u(x,t)\right],\ x>0,\ t\in[0,T]$$

when this equation has the most outstanding biological and physical significance.

1. INTRODUTION

The study of uniqueneses classes and the classes where Cauchy's Problem for partial differential systems and equations with Bessel type differential operators remain well posed was initiated by Zitomirskii [18] in 1955. This author obtains a uniqueness class for such systems between the classical functions with temperen growth when he takes into consideration partial differential equations which contain the operator $\triangle_\mu = \frac{d^2}{dx^2}+\frac{2\mu+1}{x}\frac{d}{dx}$. Later the research of intial-value problems for systems with the operator $\triangle_\mu$ has been dealt with Kiiprijanov-Kononenko [10]; Krehivshiv-Mattiuck [11]; Brzezinski [2]; among other authors.

W. Lee [12] was the first author who realized a study with a methodology similar to the one carried out by Gelfand-Shilov in the study of the systems with polynomial coefficients with the operator $\frac{d}{dx}$ (usual derivation) for systems with the differential Bessel's operator $S_\mu = \frac{d^2}{dx^2} - \frac{4\mu^2-1}{4x^2}$, $\mu > -1/2$. He obtains uniqueness classes and classes where the solutions of Cauchy's problem exist among the regular functions with compact support in the interval $I =]\,0, \infty\,[$. In 1985 the author (Gonzalez [8]) generalizes the Lee's theorems and the existence and uniqueness of such solutions were found between the functions with exponential growth, being that this growth must be associated with the type of each system which would be considered.

In this paper the operator S_μ will be studied into Hilbert spaces, which will be denoted by H_μ^q, $\mu \in R$, $q \in N$; being that each space H_μ^q is contained in the classical space $L_2(0,\infty)$ S_μ is the infinitesimal generator of a semigroup of bounded $T_\mu(t)$, which will be completely characterized. A convolution defined between measurable functions, which was defined by the author (Gonzalez [7]) will allow us to find an explicit expression of $T_\mu(t)$. Then it will be proved that the semigroup $T_\mu(t)$ is an analytic semigroup of bounded linear opdrators. The previous theorems about S_μ and $T_\mu(t)$ will lead us to the study of Cauchy's problem for an equation of parabolic type which contains the operator S_μ.

Other partial differential equations with different Bessel type operators can be solved likewise with the help of the theorems which will be obtained for the operator S_μ. Certain of these equations will be studied in the last paragraph of the paper, since they appear as significant cases of the Fokker-Planck equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left[A(x)\,\frac{\partial}{\partial x}[u(x,t)]\right] + \frac{\partial}{\partial x}\left[B(x)\;u(x,t)\right] \quad \ldots (1.1)$$

for $x \in I$, $t \in 0,T$, and $A(x)$ and $B(x)$ being coefficients which are always known.

2. SPACES H_μ^q AND L^q, $q \in N$

We denote by X the Hilbert space $L_2(0,\infty)$ with the inner product:

$$(\phi/\Phi) = \int_0^\infty \phi(x)\,\Phi(x)\,dx \;\forall\; \phi,\Phi \in X \quad \ldots (2.1)$$

The classical Hankel transformation h_μ, which is defined by the pair (2.2):

$$(h_\mu\phi)(\sigma) = \int_0^\infty (x\sigma)^{1/2} J_\mu(x\sigma)\,\Phi(x)\,dx$$

$$(h_\mu^{-1}\Phi)(x) = \int_0^\infty (x\sigma)^{1/2} J_\mu(x)\,\Phi(\sigma)\,d\sigma;\; x,\sigma \in I \quad \ldots (2.2)$$

verifies in X the following theorem 1 :

Theorem 1. (Bochner [1]) The linear operator h_μ establishes an automorphism on X.

We must understand the formulation of the previous theorem as follows:

(a) Each function $F_n(\sigma) = \int_0^n (x\sigma)^{1/2} J_\mu(x\sigma), f(x)$, $\forall\; n \in N$ belongs to the space X, being $f \in X$. Moreover the sequence $\{F_n(\sigma)\}_{n \in N}$ converges in the norm of X, being $\lim\limits_{n\to\infty} F_n(\sigma) = \int_0^\infty (x\sigma)^{1/2} J_\mu(x\sigma)\, f(x)dx$

(b) $\int_0^\infty |(h_\mu f)(\sigma)|^2\, d\sigma = \int_0^\infty |f(x)|^2\, dx$

(c) $f(x) = h_\mu\left[\,(h_\mu f)(\sigma)\,\right](x)$, $\forall\; f \in X$

For each real number μ and each integer q, we define a Hilbert space H_{μ}^{q} as follows. A function $\phi(x)$ is in H_{μ}^{q} if and only if it is defined on the interval I, it is real-valued and smooth and, for each nonnegative integer $r < q$ the expression:

$[N_{\mu+r-1}. N_{\mu+r-2} \ldots N_{\mu}(\phi)](x)$ belongs to X; being N_{μ} the differential operator:

$$(N_{\mu}\phi)(x) = x^{\mu+1/2} \frac{d}{dx}\left[x^{-\mu-1/2} \phi(x) \right], \quad x>0$$

H_{μ}^{q} is a liner space. Also we can define the norm (2.3):

$$\|\phi\|_{\mu}^{q} = \sum_{r=1}^{q} \left[\int_0^{\infty} |(N_{\mu+r-1}. N_{\mu+r-2} \ldots (N_{\mu})(\phi(x))|^2 \, dx \right]^{1/2} \tag{2.3}$$

Thus, we can prove the following lemma:

Lemma 1. The space H_{μ}^{q} is dense in X, for any $q \in N$ and $\mu>0$.

Proof: The lemma will be proved on seeing that the Fréchet space H_{μ} which was introduced by Zemanian in the form:

$H_{\mu} = \{\phi: I \to C \,/\phi$ is a smooth function, being that the expressions:

$\sup\limits_{x \in I} \left| x^m \left(x^{-1}\frac{d}{dx}\left[x^{-\mu-1/2} \phi(x) \right]\right| \right.$ remain bounded $\}$,

it is contained into the space H_{μ}^{q} $\forall q \in N$ because the composition of differential operators:

$[N_{\mu-r-1}. N_{\mu+r-2} \ldots N_{\mu} \phi(x)]$

coincides with the expression:

$$x^{\mu+r-1/2}\left[x^{-1} \frac{d}{dx} \right]^r \left[x^{-\mu-1/2} \phi(x) \right], \quad \forall \phi \in H_{\mu}$$

which appears inside the definition of H_{μ}.

Furthermore, the space of smooth functions with compact $D(I)$ is dense in X as:

$$D(I) \subset H_\mu \subset \overset{q}{H}_\mu \subset X$$

we will have proved that $\overset{q}{H}_\mu$ is dense in X.

We must now define the space L^q for $q \in N$. A function ϕ: $I \to C$ belongs to the space L^q if and only if, for any nonnegative integer r: $r \leqslant q$. the fpnction $\sigma^r \phi(\sigma)$ belongs to X.

L^q is a normed space when the norm (2.4) would be defined:

$$\Gamma^q(\phi) = \sum_{k=1}^{q} \left[\int_0^\infty |\sigma^r \phi(\sigma)|^2 d\sigma \right]^{1/2} \qquad \dots (2.4)$$

Then, we can prove the following theorem:

Theorem 2. The linear operator h_μ: $\overset{q}{H}_\mu \to L^q$; h_μ: $\phi \to h_\mu(\phi)$ defines an isometric isomorphism from the space $\overset{q}{H}_\mu$ onto the space L^q, for $\mu > -1/2$

Proof: (a) Theorem 1 proves this theorem when $q = 0$

(b) Being $q > 0$, $q \in N$ and $\phi \in \overset{q}{H}_\mu$ the formulae:

$$\sigma^r \left[h_\mu' \phi) \right] (\sigma) = (-1)^r h_{\mu+r} \left[(N_{\mu+r-1} \cdot N_{\mu+r-2} \dots N_\mu)(\phi) \right] \sigma$$

easily can be proved for any integer $r \leqslant q$ and $\sigma \in I$ and for when the recurrence relations of the Bessel function $J_\mu(z)$ are known. Then. we can use Theorem 1, and in this way we will find that $h_\mu \phi$ belongs to L^q, h_μ being an isometry.

(c) When we choose a function Φ belonging to the space L^q, $q > 0$,we know that:

$$\left[N_{\mu+r-1}\cdot N_{\mu+r-2}\cdots N_{\mu}\right]\left[h_{\mu}\Phi\right](\sigma)=(-1)^{r}\,h_{\mu+r}\left[\sigma^{r}\,\Phi(\sigma)\right](x)$$

for any integer $r \leqslant q$, *i. e.*, the function $\left[N_{\mu+r-1}\cdot N_{\mu+r-2}\cdots N_{\mu}\right]\left[h_{\mu}\Phi\right](\sigma)$ belongs to the space X, and in consequence the function $(h_{\mu}\Phi)(x)$ belongs to the space H_{μ}^{q}.

Therefore, we also have that the inverse mapping h_{μ}^{-1} coincides with h_{μ}.

3. SEMIGROUP OF BOUNDED OPERATORS $T_{\mu}(t)$

In the Hilbert space X we define the operator A associated with the differential operator S_{μ} as follows:

A is the operator:

A: $D(A) \subset X \to X$

$$\phi \to A(\phi)(x) = S_{\mu}\phi(x) = x^{-\mu-1/2}\frac{d}{dx}\,x^{2\mu+1}\frac{d}{dx}\,x^{-\mu-1/2}\,\phi(x) \quad \ldots (3.1)$$

$D(A)$ being the domain of A, given in

$$D(A) = \left\{ \phi \in X/ \text{ the function } S_{\mu}\phi(x) = \frac{d^2\phi}{dx^2} - \frac{4\mu^2-1}{4x^2}\,\phi(x) \text{ is defined, being that } S_{\mu}\phi \text{ belongs to the space } X \right\}.$$

A characterization of the domain $D(A)$ was obtained by Callias (3) in the from: When $(A'\phi)(x)$ denotes the operator: $(A'\phi)(x) = -\frac{d^2\phi(x)}{dx^2} + \frac{K}{x^2} - \phi(x)$, K being a real number: $K \geqslant 3/4$ if a function $\phi \in X$ be in the domain of the closure of the operator A' defined on $C_0^{\infty}(0,\infty)$; extending ϕ to R_- by $\phi=0$ in R_-, then we have: $\phi \in H^1(R) = \{\phi \in L_2(R) / (1+\sigma^2)^{1/2}\,\tilde{\phi}(\sigma) \in L_2(R)\}$, ϕ' is absolutely continuous on I and $\lim_{x\to 0} \phi(x) = 0$ ($\tilde{\phi}$ is the Fourier transform of ϕ).

The space H_μ^q is contained in $D(A)$ for $q>2$, since:

$$(S_\mu\phi)(x) = x^{-\mu-1/2}\frac{d}{dx}x^{2\mu+1}\frac{d}{dx}\ x^{-\mu-1/2}\ \phi(x)) = \Big[2(\mu+1)N_\mu + N_{\mu+1}\cdot N_\mu\Big]$$
$$(\phi(x)) \quad \dots (3.2)$$

then, as a consequence of (3.2), we can prove the following propertics:

(a) $D(A)$ is dense in X

(b) $-A$ is a monotone operator, because:

$$(-A\phi/\phi = -\int_0^\infty (x^{-\mu-1/2}\frac{d}{dx}x^{2\mu+1}\frac{d}{dx}\Big[x^{-\mu-1/2}(\phi(x))\Big]\phi(x)\,dx =$$

$$= \int_0^\infty |N_\mu\phi|^2\,dx \geqslant 0 \text{ for } \phi \in D(A) \quad \dots (3.3)$$

(c) A is closed operator

We want to see which is the resolvent set $\rho(A)$ of the operator A, i. e.:

$$\rho(A) = \{(\lambda \in C/\ \lambda I - A) \text{ is a bounded linear in } X\}$$

Then, let f be a function of X. We must find a function g in $D(A)$ being that:

$$(\lambda I - A)(g) = f \quad \dots (3.4)$$

Firstly, we consider that f belongs to the space $D I$; and then, applying h_μ to the two members of (3.4) we will obtain:

$$(h_\mu g)(\sigma) = \frac{1}{\sigma^2+\lambda}\ (h_\mu f)(\sigma)$$

Thus, the function $g(x)$ can be expressed in the form:

$$g(x) = h_\mu\Big[\frac{1}{\sigma^2+\lambda}(h_\mu f)(\sigma)\Big](\sigma\to x) \quad \dots (3.5)$$

We can simplify the expression (3.5) of g with the help of a convolution product which was studied by the author (González [7]). Its definition is given in the following:

Given f and g two measurable functions in I, the convolution $f{}^*g$ between f and g is defined by:

$$f{}^*g)(x) = x^{\mu+1/2} \int_0^\infty \overset{v}{\underset{x}{T}} \left[u^{-\mu-1/2}\, g(u)\right] f(y)\, y^{\mu+1/2}\, dy, \quad x > 0 \qquad \dots (3.6)$$

where $\overset{v}{\underset{x}{T}}$ denotes the translation operator:

$$\overset{v}{\underset{x}{T}} g = \frac{1}{\sqrt{\pi 2^{\mu}\Gamma(\mu+1/2)}} \int_0^\pi g([x^2 + y^2 - 2xy\, cos\omega]^{1/2})\, sin^{2\mu}\omega\, d\omega \quad \dots (3.7)$$

We can prove

Lemma 2. When the functions f and g verify:

$y^{\mu+1/2} f(y),\ y^{\mu+1/2} g(y) \in L_1(0,\infty)$, then the convolution $f{}^*g$ belongs to $L_1(0,\infty)$, too, moreover:

$$h_\mu(f{}^*g)(\sigma) = \sigma^{-\mu-1/2}\,(h_\mu f)(\sigma).\ (h_\mu g)(\sigma),\quad \sigma \in I \qquad \dots (3.8)$$

As a consequence of Lemma 2, we find that $g(x)$ in (3.5) admits the representation [14, p. 108]:

$$g(x) = h_\mu \left[\, \sigma^{-\mu-1/2}\, h_\mu \left[\lambda^{\mu/2}\, y\, K_\mu(\lambda y)\right] (h_\mu f)(\sigma) \,\right](x)$$

where K_μ is the modified Bessel function of the third class and order μ. And then:

$$g(x) = \left[\lambda^{\mu/2} f(y) \; {}^* \; y\, K_\mu(\lambda y)\right](x)$$

Thus [16, p. 367]:

$$\overset{v}{\underset{x}{T}}\left[\, u^{-\mu-1/2}\, u\, K_\mu(\lambda u) \,\right] = \lambda^{\mu/2}\, \lambda^{-\mu/2}\, x^{-\mu/2}\, \lambda^{-\mu/2}\, y^{-\mu/2}\, K_\mu(\lambda x)\, J_\mu(\lambda y) \quad (3.9)$$

Then, the function $g(x)$ which is the solution of the resolvent equation will be:

$$g(x) = \sqrt{x}\, K_\mu\,(\sqrt{\lambda}x) \int_0^\infty f(y)\, \sqrt{y}\, J_\mu\,(\sqrt{\lambda}y)\, dy =$$

$$= \sqrt{x}\, K_\mu\,(\sqrt{\lambda}x)\, \lambda^{-1/4} h_\mu\,[f(y)]\,(\lambda) \qquad \dots (3.10)$$

Moreover, the composition of the operators $N_{\mu+1}$. and N_μ give us:

$$N_{\mu+1}.\, N_\mu\,(g(x)\,) = N_{\mu+1}.\, N_\mu\, R(\lambda{:}A)\,(f) = \lambda\, R\,\lambda{:}A)\,(f)$$

such that, then, we can prove that the searched function $R\,(\lambda{:}A)(f)$ belongs to the domain $D(A)$ since the norm:

$$\|\, R(\lambda{:}A)(\,f\,)\,\|_\mu^2 \leqslant \int_0^\infty |\sqrt{x}\, K_{\mu+2}\,(\sqrt{\lambda}x)\,|^2 \left[\int_0^\infty |\,f(y)\,|^2\, dy\right.$$

$$\left[\int_0^\infty |\, J_\mu\,(\sqrt{\lambda}t)\,\sqrt{t}\,|^2\, dt\,\right]\Bigg]\, dx$$

remains bounded:

Until now, the function f had been chosen between the regular functions with bounded support in I. When f belongs to the space X the previous results can be proved too; and, in addition it is possible to obtain a constant $M > 0$, such that the norm of $R(\lambda{:}A)$ verifies:

$$\|\, R(\lambda{:}A)\,\| \leqslant M/\lambda, \quad \forall\ \lambda \in \rho(A) \qquad \dots (3.11)$$

Then we can prove

Theorem 4. The differential operator S_μ is the infinitesimal generator of a C_0 semigroup $T_\mu(t)$.

Following the proof of Theorem 4 we know (Pazy [15]) that the semigroup $T_\mu(t)$ can admit the following representation:

$$T_\mu(t)(g)\,(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda t} \left[R(\lambda{:}A)g \right](x)\, d\lambda =$$

$$= \int_0^\infty g(y)\, \sqrt{y}\,(-i)^\mu\, L^{-1} \left[\, K_\mu\,(\sqrt{\lambda}x)\, I_\mu\,(-\sqrt{\lambda}yi)\,\right](t)\, dy$$

where g belongs to X and L^{-1} denotes the inverse Laplace transformation. However, the inverse transformation of $K_\mu(\sqrt{\lambda}x)I_\mu(-\sqrt{\lambda}\, yi)$ is known

(Ditkin-Proudnikov [4,p. 344], so we can obtain the expression of $T_\mu(t)$ given in (3.12):

$$[T_\mu(t)g](x) = x(2t)^{-1}\left[e^{-x^2/4t}\int_0^\infty g(y)\sqrt{y}\,J_\mu(xy/2t)\,e^{y^2/4t}\,dy + \right.$$

$$\left. + \; e^{-x^2/4t}\int_0^\infty g(y)\sqrt{y}\,J_\mu(xy/2t)\,e^{-y^2/4t}\,dy\right] \quad \dots (3.12)$$

Moreover, if we say that the semigroup $T_\mu(t)$ is analytic when there exist two nonnegative constants $\triangle$ and C and

$$\| A\, R(\lambda:A)^{n+1} \| \leqslant C/n\lambda^n, \; \forall\, n \in N \text{ and } \forall\, \lambda \geqslant n\triangle \quad \dots (3.13)$$

then

$$[R(\lambda:A)^2 f](x = \sqrt{x}K_\mu(\lambda x)\int_0^\infty J_\mu(\sqrt{\lambda}y)\sqrt{y}\, f(y)dy$$

$$\left[\int_0^\infty \sqrt{u}K_\mu(\sqrt{\lambda}u\, J_\mu(\sqrt{\lambda}u)\; du\right] = (2\lambda)^{-1}\,[R(\lambda:A)f](x)$$

it will be:

$$[R(\lambda:A)^{n+1}f](x) = (2\lambda)^{-n}\,[R(\lambda:A)f](x) \quad \forall\, n \in N \quad \dots (3.14)$$

and:

$$[AR(\lambda:A)^{n+1}f](x) = 2^{-n}\,\lambda^{1-n}\,[R(\lambda:A)f](x)$$

being, then that the following theorem has been proved:

Theorem 5. The operator S_μ is the infinitesimal generator of an analytic semigroup $T_\mu(t)$, given as the formula (3.12)

4. CAUCHY'S PROBLEM FOR A PARTIAL DIFFERENTIAL EQUATION OF PARABOLIC TYPE WITH THE OPERATOR S_μ.

let us consider the following partial differential equation:

$$\frac{\partial u}{\partial t}(x,t) = \left[\frac{d^2}{dx^2} - \frac{4\mu^2-1}{4x^2}\right](u(x,t)),\ x \in I,\ t \geqslant 0 \qquad \dots (4.1)$$

with the intial values condition:

$$u(x,0) = u_0(x) \qquad \dots (4.2)$$

When the solution $u(x,t)$ of (4.1) and (4.2) would be understood as a abstract function:

$$u.]\,0,\infty\,[\to X$$
$$t \to u(t): I \to R$$
$$x \to u(t)(x) = u(x,t)$$

the semigroups of linear operators theory can be used in the resolution of the partial diferential equations and then we can prove the theorem 6.

Theorem 6. When $u_0(x)$ is a function which belongs to the space X, then there is an unique solution $u(x,t)$ of (4.1) and (4.2) in the space X.

Moreover, if Problems (4.1) and (4.2) are inhomogeneous in the form:

$$\frac{\partial u}{\partial t}(x,t) = S_\mu[u(x,t)] + f(x,t) \qquad \dots (4.3)$$

$$u(x,0) = u_0(x),\ x \in I,\ 0 \leqslant t \leqslant T < \infty \qquad \dots (4.4)$$

we will have

Theorem 7. If the function f belongs to the space $L_1(0,T,X)$ and it is locally Holder continuous in $[0,T]$, for each $u_0(x)$ belonging to X, there is a unique solution $u(x,t)$ of (4.3) and (4.4) which admits the expression:

$$u(x\ t) = T_\mu(t)\ [u_0(x)] + \int_0^t T_\mu'(t-s(\ [f(x,s)]\ ds$$

Theorems 6 and 7 can be modified and then they may be used in the study of other parabolic equations of type (4.1)-(4.2) where these appear different Bessel-type differential operators of S_μ. These operators which present a general expression in the form $x^a \frac{d}{dx}\ x^b\ \frac{d}{dx}\ x^c; a+b+c < 2$ have been studied by a group of research workers of the Mathematical Analysis Department of the University of La Laguna (Canaary Islands). They have found every one of the topics relative to the operational treatment of those operators; i.e. they have found integral transformations. spaces of test functions where these operators define linear and continuous mappings; convolution product associated with them; etc. In particular, the Bessel-Clifford's operator $B_\mu = x.\ \frac{d^2}{dx^2} + (1-\mu)\frac{d}{dx} = \frac{d}{dx}\ x^{\mu+1}\ \frac{d}{dx}\ x^{-\mu}$, $\mu > 0$ has been carefully analysed (González, [7]).

We can define

$$\frac{\partial u}{\partial t}(x,t) = B_\mu\ [u(x,t)] \quad x > 0\ \ t \geqslant 0 \qquad \text{... (4.5)}$$

being the initial value problems (4.5) and (4.6):

$$u(x,0) = u_0(x) \qquad \text{... (4 6)}$$

could be studied with the help of the previous theorems.

So, when $u(x,t)$ is a function which belongs ($\forall\ t \geqslant 0$) to the space

$$X' = \left\{ f: I \to R \,/ \int_0^\infty |f(u)|^2\, u^{-\mu}\ du < \infty \right\}$$

which is a normed space with the norm:

$$\|f\|_{\mu,2} = \left[\int_0^\infty |f(u)|^2\, u^{-\mu}\ du \right]^{1/2} f \in X'$$

then the function:

$$v(x,t) = 2^{-1} x^{1/2-\mu}\ u(x^2/2,t)$$

will belong to the space X in 2 , $\forall\ t \geqslant 0$. Moreover, we can formally find the relation:

$$S_\mu[x^{-\mu+1/2}\ u(x^2/2,t)] = x^{-\mu+1/2}\ B_\mu\ [u(x,t)]\ (x^2/2),\ \forall\ x \in I,\quad t > 0$$

and then, when $v(x,t)$ could be a solution of (4.1) - (4.2) defined into the space X, $u_0(x)$ belonxing to the space X; $u(x,t)$ will be a solution in X' of (4.5) and (4.6) for $v_0(x)$ chosen in the space X'. In this way, the initial value problem (4.5) and (4.6) can be solved, and then two theorems like the 6 and 7 ones are thus proven.

The previous operational treatment of parabolic equations can also be used likewise in the study of equations of the same type with other Bessel type differential operators; among others the following ones:

$$x^{-\mu}\frac{d}{dx}\,x^{\mu+1}\frac{d}{dx} = x\,\frac{d^2}{dx^2} + (\mu+1)\,\frac{d}{dx} = \frac{d}{dx}\left[x\,\frac{d}{dx}\right] + \mu\frac{d}{dx}$$

or

$$x^{-2\mu-1}\frac{d}{dx}\,x^{2\mu+1}\frac{d}{dx} = \frac{d^2}{dx^2} + \frac{2\mu+1}{x}\,\frac{d}{dx}\,.$$

These differential operators usually appear in particular cases of Fokker Planck equation (1.1); precisely in those cases where the said equation has the most outstanding biological and physical significance (González, [9]).

ACKNOWLEDGEMENTS

The author wishes to express deep sense of gratitude to Professor H. M. Srivastava, University of Victoria, for his kind and valuable suggestions to bring the manuscript in the present form.

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