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A FIXD POINT THEOREM FOR A PAIR OF EXPANSIVE MAPS

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In a recent paper the author [1] established several fixed point theorems for a pair of surjective expansive maps. In this note we prove a theorem for a pair of expansive maps, and demonstrate that the first two results of [1] follow as special cases.

Theorem. *Let f,g be surjective selfmaps of a complete metric space (X,d). Suppose there exists a constant $a>1$ such that*

(1) $d(fx,gy) \geqslant a\, d(x,y)$

for each x,y in $\{x_n\}$ satisfying either $x \in f^{-1}y$ or $y \in g^{-1}x$, *where $\{x_n\}$ is defined by $x_0 \in X$, $x_{2n+1} \in f^{-1}x_{2n}$, $x_{2n+2} \in g^{-1}x_{2n+1}$. Then f and g have a unique common fixed point.*

Proof. The proof of Theorem 1 of [1] only involves points in the closure of $\{x_n\}$. Therefore that proof applies, unchanged, to this theorem.

Corollary 1. ([1, Theorem 1]). *Let f,g be surjective selfmaps of a complete metric space (X,d). Suppose there exists a constant $a>1$ such that*

(2) $d(fx,gy) \geqslant a\, d(x,y)$

for each x,y in X. Then f and g have a unique common fixed point.

Since (2) is true for all points x,y in X, it is surely true for those points in the closure of $\{x_n\}$,

Corollary 2. ([1, Theorem 2]). *Let f,g be surjective selfmaps of a complete metric space (X d). Suppose there exists nonnegative functions* p,q,r,s,t *satisfying*

(3) $\inf_{x,y \in X} (p(x,y) + q(x,y) + t(x,y)) > 1,$

(4) $\inf_{x,y \in X} \{1-q(x,y) + r(x,y), (1-p(x,y) + s(x,y))\} > 0,$

(5) $\sup_{x,y \in X} \{p(x,y), q(x,y)\} > 1,$

and

(6) $d(fx,gy) \geqslant p(x,y)\, d(x,fx) + q(x,y)\, d(y,gy) + r(x,y)\, d(x,gy)$
$+ s(x,y)\,(dy,fx) + t(x,y)\, d(x,y)$

for all x,y in X, x=y. Then f and g have a untque common flxed point..

Proof. Set $y = fx$ in (6) to get

$$d(fx,gfx) \geqslant pd(x,fx) + qd(fx,gfx) + rd(x,gfx) + td(x,fx)$$
$$\geqslant pd(x,fx) + qd(fx,gfx) + r[d\, x,fx) - d(fx,gfx)\,] + td(x,fx),$$

where p,q,r,s,t are evaluated at (x,fx). We then obtain

$$d(fx,gfx) \geqslant ad(x,fx),$$

where

$$a = \inf_{x\, y \in X} \frac{p+t+r}{1-q+r} > 1$$

from (2).

REFERENCE

[1] B. E. Rhoades, Some fixed point theorems for pairs of mappings. *Jñānābha* **15** (1985), 151–156.