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IDEAL IN SEMIPRIME NONASSOCIATIVE RINGS

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ABSTRACT

The following theorem is proved: Let R be a semiprime non-associative ring. If A is a right (or left) indeal of R contained in the nucleus and the associator ideal of R, then $A = (0)$.

1. INTRODUCTION

A non-associative ring does not presuppose the associative law of multiplication. The nucleus N of an arbitrary non-associative ring R consists of all elements x in R such that

$$(x,y,z) = 0 = (y,z,x) = (z,x,y) \quad \text{for all } y,z \text{ in } R$$

where the associator (x,y,z) is defined by

$$(x,y,z) = (xy)\, z - x\,(yz).$$

It is well known (see Schafer [3], p. 13) than N is an associative subring of R. All rings R have an ideal, called the associator ideal. It is defined as the smallest ideal which contains all associators, It actually consists of all finite sums of associators and right or left multiples of associators (see Kleinfeld [1], [2]). The associator ideal is never zero, except when R is associative. A ring R is called semiprime if R has no nonzero ideal squaring to zero. A straightforward verification shows that any ring satisfies

$$(wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z,$$

which is known as the Teichüller identity.

2. THE MAIN RESULT

we prove the following:

Theorem. Let R be a semiprime nonassociative ring. If A is a right (or left) idel of R contained in the nucleus and the associator ideal of R, then $A = (0)$.

Proof. We shall use the symbol R (or A) to represent an element of R (or A). When this notation is used, we mean to consider not only all possible expressions where $R's$ (or $A's$) are replaced by elements of R (or A), but the additive subgroup generated by these. Let $I = (R,R,R) + (R,R,R)\,R$ be the associator ineal of R. Now $A \subseteq N$ implies that $(A,R,R) = (R,A,R) = (R,R,A) = (0)$. For any a in A and x,y,z in R, by the Teichmüller identity,

$(ax,y,z) - (a,xy,z) + (a,x,yz) - a(x,y,z) - (a,x,y)\,z = 0$. If A is a right ideal, (ax,y,z), (a,xy,z), (a,x,yz), $(a,x,y) \in (A,R,R) = (0)$. Therefore, $a(x,y,x) = 0$. Since $A \subset N$, we have $A\,((R,R,R)R) = (A(R,R,R)R) = (0)$. This implies that $A((R,R,R)+(R,R,R)R) = (0)$. Thus, $AI = (0)$. Now $R(A + RA) \subseteq RA + R(RA) \subseteq RA + (RR)A - (R,R,A) \subseteq A + RA$. Also $(A+RA)\,R \subseteq AR + (RA)R \subseteq A + R\,(AR) \subseteq A+RA$. Thus if $<A>$ is the ideal generated by A then $A+RA = <A>$. Therefore, $<A>\ I = (A+RA)\,I \subseteq AI + (RA)I \subseteq R\,(AI) \subseteq (0)$. Moreover $A \subseteq I$ and I is an ideal implies that $<A> \subseteq I$. Thus $<A><A> = 0$, New let $I = (R,R,R) + R(R,R,R)$ be the associator ideal of R. Again for any a in A and $w\ x,y$ in R, by the Teicnmüller identity, $(wx,y,a) - (w,xy,a)$, $(w,x,y,a) - w(x,y,a) - (w,x,y)\,a = 0$. If A is a left ideal, (wx,y,a), (w,xy,a), (w,x,ya), $w(x,y,a) \in (R,R,A) = (0)$. Therefore, $(w,x,y)\,a = (0)$. Sinee $A \subseteq N$, we have $(R(R,R,R)\,A = R((R,R,R)\,A) = (0)$. This implies that $((R,R,R)\,R + R(R,R,R))\,A = (0)$. Thus

$((R,R,R)R + R(R,R,R))A = (0)$. Thus, $IA = (0)$. Now $(A + AR)R \subseteq AR + (AR)R \subseteq AR + A(RR) + (A,R,R) \subseteq A + AR$. Also $R(R+AR) \subseteq RA + R(AR) \subseteq A + (RA)R \subseteq A+AR$. Thus if $<A>$ is the ideal generated by A, then $A+AR = <A>$. Therefore, $I <A> = I(A+AR) \subseteq IA + I(AR) \subseteq (IA)R \subseteq (0)$. Moreover $A \subseteq I$ and I is an ideaal implies that $<A> \subseteq I$. Thus $<A> <A> = (0)$.

We have shown that if A is a right (or left) ideal of R then $<A> <A> = (0)$. But R is semi-prime and $<A>$ is an ideal of R. This implies that $<A> = (0)$. Hence $A = (0)$.

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