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Fractional Integration and Integral Representations of Karlsson's Multiple Hypergeometric Function and its Confluent Forms

by

R. C. SINGH CHANDEL AND P. K. VISHWAKARMA
Department of Mathemetics, D. V. Postgraduate College, Orai–285001, U. P., India

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1. INTRODUCTION

Joshi [6] made use of the theory of fractional integration in order to deduce Eulerian Integral representations of hypergeometric functions of three variables, viz. H_A, H_B and H_C defined and studied by Srivastava (*cf* [10] to [14] see also [7] and [15]). In subsequent paper, Chandel [1], obtained simllar integral reprecentations for the multiple hypergeometric functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, $F_D^{(n)}$ of Lauricella [9] and also for their confluent forms $\Phi_2^{(n)}$ and $\psi_2^{(n)}$. Further in another paper. Chandel [2] deduced similar integral representations for Exton's multiple hypergeometric functions ${}^{(k)}_{(1)}E_D^{(n)}$, ${}^{(k)}_{(2)}E_D^{(n)}$ [5] related to Lauricella's $F_D^{(n)}$ and also for his own multiple hypergeometric function ${}^{(k)}_{(1)}E_C^{(n)}$ related to Lauric lla's $F_C^{(n)}$. Recently, Chandel and Gupta [3] have defined and studied multiple hypergeometric functions ${}^{(k)}F_{AC}^{(n)}$, ${}^{(k)}F_{AD}^{(n)}$, ${}^{(k)}F_{BD}^{(n)}$ related to Lauricella's functions [9]. Motivated by this paper karlsson [8], also introduced fourth possible intermediate Lauricella function ${}^{(k)}F_{CD}^{(n)}$. Very recently the author Chandel and Vishwakarma [4] have studied confluent forms ${}^{(k)}_{(2)}\Phi_{CD}^{(n)}$, ${}^{(k)}_{(3)}\Phi_{CD}^{(n)}$ ${}^{(k)}_{(4)}\Phi_{CD}^{(n)}$ of ${}^{(k)}F_{CD}^{(n)}$.

In the present paper, we shall make use of the theory of fractional integration to derive Eulerian integral representation for multiple hypergeometric function ${}^{(k)}F_{CD}^{(n)}$ of Karlsson [8] and for its confluent forms ${}^{(k)}_{(2)}\Phi_{CD}^{(n)}$,

$\overset{(k)}{(3)}\Phi_{CD}^{(n)}$ and $\overset{(k)}{(4)}\Phi_{CD}^{(n)}$ introduced by the authors [4].

We recall that the rule for fractional integration by parts is given by

$$\int_a^b U\frac{\partial^v V}{\partial(b-x)^v}\,dx=\int_a^b V\frac{\partial^v U}{\partial(x-a)^v}\,dx. \tag{1.1}$$

If Re $(\nu)>0$, the fractional derivatives occuring in (1.1) are defined by the following integrals :

$$\frac{\partial^\nu U}{\partial(x-a)^v}=\frac{1}{\Gamma(-v)}\int_a^x (x-y)^{-v-1}\,U(y)\,dy. \tag{1.2}$$

$$\frac{\partial^v V}{\partial(b-x)^v}=\frac{1}{\Gamma(-v)}\int_x^b (y-x)^{-v-1}\,V(v)\,dy. \tag{1.3}$$

If Re $(\nu)>0$ and U, V are expressible in terms of the series

$$U=\Sigma\, A_r(x-a)^{\rho+r-1},\quad V=\Sigma\, B_s(b-x)^{\rho+s-1}$$

then the fractional derivatives are obtained by differentiating the above series term by term, with the help of the formula.

$$\frac{\partial^v W^{\mu-1}}{\partial W^v}=\frac{\Gamma(\mu)}{\Gamma(\mu-\nu)}\,W^{\mu-v-1} \tag{1.4}$$

provided $\mu\neq v$.

2. THE INTEGRAL REPRESENTATIONS

In this section we derive the integral representations for the functions $\overset{(k)}{F}{}_{CD}^{(n)}$, $\overset{(k)}{\Phi}{}_{CD}^{(n)}$, $\overset{(k)}{\Phi}{}_{CD}^{(n)}$ and $\overset{(k)}{\Phi}{}_{CD}^{(n)}$.

Consider

$$\frac{\partial^{\beta_1+\nu_1+\ldots+\beta_k-\nu_k}}{\partial x_1{}^{\beta_1-\nu_1}\ldots\partial x_k{}^{\beta_k-\nu_k}}\left\{x_1{}^{\beta_1-1}\ldots x_k{}^{\beta_k-1}\right.$$

$$\left.\overset{(k)}{F}{}_{CD}^{(n)}[\alpha,\ \nu,\ \nu_1,\ \ldots,\ \nu_k;\ \gamma,\ \gamma_{k+1},\ \ldots,\ \gamma_n;\ x_1,\ \ldots,\ x_n]\right\}$$

$$=\sum_{m_1,\ \ldots,\ m_n=0}^{\infty}\frac{(\alpha)_{m_1+\ldots m_n}\,(\nu)_{m_{k+1}+\ldots+m_n}\,(\nu_n)_{m_1}\ldots(\nu_k)_{m_k}}{(\gamma)_{m_1+\ldots+m_k}\,(\gamma_{k+1})_{m_{k+1}}\ldots(\gamma_n)_{m_n}}\cdot\frac{1}{m_1!\ldots m_n!}$$

$$\frac{\partial^{\beta_1-\nu_1+\ldots+\beta_k-\nu_k}}{\partial x_1{}^{\beta_1-\nu_1}\ldots\partial x_k{}^{\beta_k-\nu_k}}\left\{x_1{}^{\beta_1+m_1-1}\ldots x_k{}^{\beta_k-m_k-1}\right.$$

$$\left.x_{k+1}^{m_{k+1}}\ldots x_n^{m_n}\right\}$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\beta_j)}{\Gamma(\nu_j)} (x_j)^{\nu_j - 1} \, {}^{(k)}F_{CD}^{(n)} [\alpha, \nu, \beta_1, \ldots, \beta_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1, \ldots, x_n]$$

Hence by using the relation (1.2), we have

$$ {}^{(k)}F_{CD}^{(n)} [\alpha, \nu, \beta_1, \ldots, \beta_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1, \ldots, x_n]$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\nu_j)}{\Gamma(\beta_j)} (x_j)^{1-\nu_j} \quad \frac{\partial^{\beta_1 - \nu_1 + \ldots + \beta_k - \nu_k}}{\partial x_1^{\beta_1 - \nu_1} \ldots \partial x_k^{\beta_k - \nu_k}}$$

$$\left\{ x_1^{\beta_1 - 1} \ldots x_k^{\beta_k - 1} \right.$$

$$\left. {}^{(k)}F_{CD}^{(n)} [\alpha, \nu, \nu_1, \ldots, \nu_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1, \ldots, x_n] \right\}$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\nu_j)(x_j)^{-\nu_j+1}}{\Gamma(\beta_j)\,\Gamma(\nu_j - \beta_j)} \int_0^{x_1} \ldots \int_0^{x_k} (x_1 - y_1)^{\nu_1 - \beta_1 - 1} y_1^{\beta_1 - 1}$$

$$\ldots (x_k - y_k)^{\nu_k - \beta_k - 1} y_k^{\beta_k - 1}$$

$$ {}^{(k)}F_{CD}^{(n)} [\alpha, \nu, \nu_1, \ldots, \nu_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; y_1, \ldots, y_k, x_{k+1},$$

$$\ldots, x_n] \, dy_1 \ldots dy_k.$$

Now putting every $y_j = x_j t_j$, where $j = 1, \ldots, k$,

we establish

(2 1) $$ {}^{(k)}F_{CD}^{(n)} [\alpha, \nu, \beta_1, \ldots, \beta_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1, \ldots, x_n]$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\nu_j)}{\Gamma(\beta_j)\,\Gamma(\nu_j - \beta_j)} \int_0^1 \ldots \int_0^1 \prod_{i=1}^{k} (1-t_i)^{\nu_i - \beta_i - 1} (t_i)^{\beta_i - 1}$$

$$ {}^{(k)}F_{CD}^{(n)} [\alpha, \nu, \nu_1, \ldots, \nu_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1 t_1, \ldots, x_k t_k, x_{k+1}, \ldots, x_n]$$

$$dt_1 \ldots dt_k,$$

provided $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i)$, $\quad i = 1, \ldots, k.$

For brevity we use the operator $\mathfrak{H}$ defined by chandel [1, (2.2)] :

(2.2) $$\mathfrak{H}_{\nu_1, \ldots, \nu_k}^{\beta_1, \ldots, \beta_k} \left\{ \text{[illegible]} \right\}$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\nu_j)}{\Gamma(\beta_j)\,\Gamma(\nu_j-\beta_j)} \int_0^1 \cdots \int_0^1 \prod_{i=1}^{k} (t_i)^{\beta_i-1} (1-t_i)^{\nu_i-\beta_i-1} \{\}\, dt_1 \ldots dt_k$$

where $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i)$, $i = 1, \ldots, k$.

Hence the relation (2.1) can be written as

$$(2.3)\quad \mathfrak{L}\left\{ {}^{(k)}F_{CD}^{(n)}[\alpha, \nu, \nu_1, \ldots, \nu_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1 t_1, \ldots, x_k t_k, x_{k+1}, \ldots, x_n] \right\}$$

$$= {}^{(k)}F_{CD}^{(n)}[\alpha, \nu, \beta_1, \ldots, \beta_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1 \ldots, x_n]$$

provided $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i)$, $i = 1, \ldots, k$.

Applying the same techniques, we also obtain the following operational relations.

$$(2.4)\quad \mathfrak{L}\left\{ {}^{(k)}_{(2)}\Phi_{CD}^{(n)}[\alpha, \nu_1, \ldots, \nu_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1 t_1, \ldots, x_k t_k, x_{k+1}, \ldots x_n] \right\}$$

$$= {}^{(k)}_{(2)}\Phi_{CD}^{(n)}[\alpha, \beta_1, \ldots, \beta_k\ \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1 \ldots, x_n]$$

provided $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i)$, where $i = 1, \ldots, k$.

$$(2.5)\quad \mathfrak{L}\left\{ {}^{(k)}_{(3)}\Phi_{CD}^{(n)}[\nu, \nu_1, \ldots, \nu_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1 t_1 \ldots, x_k t_k, x_{k+1}, \ldots, x_n] \right\}$$

$$= {}^{(k)}_{(3)}\Phi_{CD}^{(n)}[\nu, \beta_1, \ldots, \beta_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1, \ldots, x_n],$$

provided $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i)$, where $i = 1, \ldots, k$.

$$(2.6)\quad \mathfrak{L}\left\{ {}^{(k)}_{(4)}\Phi_{CD}^{(n)}[\alpha, \nu, \nu_1, \ldots, \nu_k; \gamma; x_1 t_1, \ldots, x_k t_k, x_{k+1}, \ldots, x_n] \right\}$$

$$= {}^{(k)}_{(4)}\Phi_{CD}^{(n)}[\alpha, \nu, \beta_1, \ldots, \beta_k; \gamma; x_1, \ldots, x_n]$$

provided $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i)$, $i = 1, \ldots, k$.

$$(2.7)\quad \mathfrak{L}\left\{ {}^{(k)}F_{CD}^{(n)}[\alpha, \nu, \gamma_1, \nu_2, \ldots, \nu_k; \beta_1, \gamma_{k+1}, \ldots, \gamma_n; x_1 t_1, x_1 x_2 t_1 t_2, \ldots, x_1 x_k t_1 t_k, x_{k+1}, \ldots, x_n] \right\}$$

$$= {}^{(k)}F_{CD}^{(n)}[\alpha, \nu, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1, \gamma_{k+1}, \ldots, \gamma_n; x_1, x_1 x_2, \ldots, x_1 x_k, x_{k+1}, \ldots, x_n],$$

provided $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i) = 1, \ldots, k$.

$$(2.8)\quad \mathfrak{L}\left\{ {}^{(k)\ (n)}_{(2)}\Phi_{CD}[\alpha, \gamma_1\ \nu_2, \ldots, \nu_k; \beta_1, \gamma_{k+1}, \ldots, \gamma_n; x_1 t_1, x_1 x_2 t_1 t_2, \ldots, x_1 x_k t_1 t_k, x_{k+1}, \ldots, x_n] \right\}$$

$$= {}^{(k)\ (n)}_{(2)}\Phi_{CD}\,[\alpha, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1, \gamma_{k+1}, \ldots, \gamma_n; x_1, x_1x_2, \ldots, x_1x_k, x_{k+1}, \ldots, x_n]$$

provided $\quad 0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i),\ i=1, \ldots, k.$

(2.9) $$\mathfrak{L}\left\{{}^{(k)\ (n)}_{(3)}\Phi_{CD}\,[\nu, \gamma_1, \nu_2, \ldots, \nu_k; \beta_1, \gamma_{k+1}, \ldots, \gamma_n; x_1t_1, x_1x_2t_1t_2, \ldots, x_1x_kt_1t_k, x_{k+1}, \ldots, x_n]\right\}$$

$$= {}^{(k)\ (n)}_{(3)}\Phi_{CD}\,[\nu, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1, \gamma_{k+1}, \ldots, \gamma_n; x_1, x_1x_2, \ldots, x_1x_k, x_{k+1}, \ldots, x_n],$$

provided $\quad 0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i),\ i=1, \ldots, k.$

(2.10) $$\mathfrak{L}\left\{{}^{(k)\ (n)}_{(4)}\Phi_{CD}\,[\alpha, \nu, \gamma_1, \nu_2, \ldots, \nu_k; \beta_1; x_1t_1, x_1t_1x_2t_2, \ldots, x_1t_1x_kt_k, x_{k+1}, \ldots, x_n]\right\}$$

$$= {}^{(k)\ (n)}_{(4)}\Phi_{CD}\,[\alpha, \nu, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1; x_1, x_1x_2, \ldots, x_1x_k, x_{k+1}, \ldots, x_n],$$

provided $\quad 0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i),\ i=1, \ldots, k.$

3. FURTHER EXTENSIONS

In this section, we derive extensions of the results obtained in the preceding section to hold for multiple hypergeometric function, defined by

$$\mathring{F}(x_1, \ldots, x_n) = F\left[\begin{array}{c|c|c} 1 & \alpha & \\ 1 & (-;\nu) & \\ k & \lambda_1, \nu_1; \ldots; \lambda_k, \nu_k, & x_1, \ldots, x_n \\ 0 & \ldots \quad \ldots & \\ 1 & (\gamma;-) & \\ n & \mu_1; \ldots, \mu_k; \gamma_{k+1}; \ldots; \gamma_n & \end{array}\right]$$

$$= \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_n}(\nu)_{m_{k+1}+\ldots+m_n}(\lambda_1)_{m_1}\ldots(\lambda_k)_{m_k}(\nu_1)_{m_1}\ldots(\nu_k)_{m_k}}{(\gamma)_{m_1+\ldots+m_k}(\mu_1)_{m_1}\ldots(\mu_k)_{m_k}(\gamma_{k+1})_{m_{k+1}}\ldots(\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}.$$

From the above defintion it follows that

$$\frac{\partial^{\mu_1-\lambda_1+\ldots+\mu_k-\lambda_k}}{\partial t_1^{\mu_1-\lambda_1}\ldots\partial t_k^{\mu_k-\lambda_k}}\left\{t_1^{\mu_1-1}\ldots t_k^{\mu_k-1}\overset{\circ}{F}(x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n)\right\}$$

$$=\prod_{j=1}^{k}\frac{\Gamma(\mu_j)}{\Gamma(\lambda_j)}(t_j)^{\lambda_j-1}\,{}^{(k)}F_{CD}^{(n)}[\alpha,\nu,\nu_1,\ldots,\nu_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n].$$

Using the relation (2.3), we have

$${}^{(k)}F_{CD}^{(n)}[\alpha,\nu,\beta_1,\ldots,\beta_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1,\ldots,x_n]$$

$$=\prod_{j=1}^{k}\frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\,\Gamma(\mu_j)\,\Gamma(\nu_j-\beta_j)}\int_0^1\ldots\int_0^1\prod_{i=1}^{k}(t_i)^{\beta_i-\lambda_i}(1-t_i)^{\nu_i-\beta_i-1}$$

$$\frac{\partial^{\mu_1-\lambda_1+\ldots+\mu_k-\lambda_k}}{\partial t_1^{\mu_1-\lambda_1}\ldots\partial t_k^{\mu_k-\lambda_k}}\left\{t_1^{\mu_1-1}\ldots t_k^{\mu_k-1}\overset{\circ}{F}(x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n)\right\}dt_1\ldots dt_n.$$

By using the relation (1.1), we further obtain

$${}^{(k)}F_{CD}^{(n)}[\alpha,\nu,\beta_1,\ldots,\beta_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1,\ldots,x_n]$$

$$=\prod_{j=1}^{k}\frac{\Gamma(\nu_j)\,\Gamma\lambda_j)}{\Gamma(\beta_j)\,\Gamma(\mu_j)\,\Gamma(\nu_j-\beta_j)}\int_0^1\ldots\int_0^1 t_1^{\mu_1-1}\ldots t_k^{\mu_k-1}$$

$$\overset{\circ}{F}(x_1t_1,\ldots,x_kt_k\,x_{k+1},\ldots,x_n)\;\frac{\partial^{\mu_1-\lambda_1+\ldots+\mu_k-\lambda_k}}{\partial(1-t_1)^{\mu_1-\lambda_1}\ldots\partial(1-t_k)^{\mu_k-\lambda_k}}$$

$$\left\{\prod_{i=1}^{k}(t_i)^{\beta_i-\lambda_i}\;(1-t_i)^{\nu_i-\beta_i-1}\right\}dt_1\ldots dt_k$$

$$=\prod_{j=1}^{k}\frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\,\Gamma(\mu_j)\,\Gamma(\nu_j+\lambda_j-\mu_j-\beta_j)}\int_0^1\ldots\int_0^1\prod_{j=1}^{k}(t_i)^{\mu_i-1}$$

$$(1-t_i)^{\mu_i+\lambda_i-\mu_i-\beta_i-1}$$

$${}_2F_1[\nu_i-\beta_i,\lambda_i-\beta_i;\nu_i+\lambda_i-\mu_i-\beta_i;(1-t_i)]\overset{\circ}{F}(x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n)\,dt_1\ldots dt_k.$$

For brevity, we use the following operator **R** introduced by Chandel [1,(3.1)] :

$$(3.1)\qquad R\left\{\ \right\}=\prod_{j=1}^{k}\frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\,\Gamma(\mu_j)\,\Gamma(\nu_j+\lambda_j-\beta_j-\mu_j)}\int_0^1\cdots\int_0^1\prod_{j=1}^{k}(t_j)^{\mu_j-1}$$

$$(1-t_i)^{\nu_i+\lambda_i-\mu_i-\beta_i-1}\ {}_2F_1\left[\nu_i-\beta_i,\ \lambda_i-\beta_i;\ \nu_i+\lambda_i-\mu_i-\beta_i;\ (1-t_i)\right]$$

$$\left\{\ \right\}\,dt_1\ldots dt_k.$$

We then find the following result

$$(3.2)\qquad R\left\{F\left[\begin{array}{c|c|c}1 & & \alpha\\ 1 & & (-;\nu)\\ k & & \lambda_1,\nu_1;\ \ldots;\ \lambda_k,\nu_k\\ (& & \ldots\quad\ldots\quad\ldots\\ 1 & & (\gamma;-)\\ n & \mu_1;\ \ldots;\ \mu_k;\ \gamma_{k+1};\ \ldots;\ \gamma_n & \end{array}\ \middle|\ x_1t_1,\ \ldots,\ x_kt_k,\ x_{k+1},\ \ldots,\ x_n\right]\right\}$$

$$=\ {}^{(k)\ (n)}_{CD}F\ [\alpha,\ \nu,\ \beta_1,\ \ldots,\ \beta_k;\ \gamma,\ \gamma_{k+1},\ \ldots,\ \gamma_n;\ x_1,\ \ldots,\ x_n],$$

provided that $0<\mathrm{Re}\,(\mu_j)<\mathrm{Re}\,(\nu_j+\lambda_j-\beta_j);\ 0<\mathrm{Re}\,(\beta_j)<\mathrm{Re}\,(\nu_j),\ j=1,\ \ldots,\ k.$

[Extension of (2.3)]

Applying the same techniques we obtain the following results :

$$(3.3)\qquad R\left\{F\left[\begin{array}{c|c|c}1 & & \alpha\\ 0 & & \ldots\quad\ldots\quad\ldots\\ k & & \lambda_1,\nu_1;\ \ldots;\ \lambda_k,\nu_k\\ 0 & & \ldots\quad\ldots\quad\ldots\\ 1 & & (\gamma;-)\\ n & \mu_1;\ \ldots;\ \mu_k;\ \gamma_{k+1};\ \ldots;\ \gamma_n & \end{array}\ \middle|\ x_1t_1,\ \ldots,\ x_kt_k,\ x_{k+1},\ \ldots,\ x_n\right]\right\}$$

$$=\ {}^{(k)\ (n)}_{(2)\ CD}\Phi\ [\alpha,\ \beta_1,\ \ldots,\ \beta_k;\ \gamma,\ \gamma_{k+1},\ \ldots,\ \gamma_n;\ x_1,\ \ldots,\ x_n]$$

valid if $0<\mathrm{Re}\,(\beta_i)<\mathrm{Re}\,(\nu_i);\ 0<\mathrm{Re}\,(\mu_i)<\mathrm{Re}\,(\nu_i+\lambda_i-\beta_i),\ i=1,\ \ldots,\ k.$

[Extension of (2.4)]

$$(3.4)\qquad R\left\{F\left[\begin{array}{c|c|c}0 & & \ldots\quad\ldots\quad\ldots\\ 1 & & (-;\nu)\\ k & & \lambda_1,\nu_1;\ \ldots;\ \lambda_k,\nu_k\\ 0 & & \ldots\quad\ldots\quad\ldots\\ 1 & & (\gamma;-)\\ n & \mu_1;\ \ldots;\ \mu_k;\ \gamma_{k+1};\ \ldots,\ \gamma_n & \end{array}\ \middle|\ x_1t_1,\ \ldots,\ x_kt_k,\ x_{k+1},\ \ldots,\ x_n\right]\right\}$$

$$=\ {}^{(k)\ (n)}_{(3)\ CD}\Phi\ [\nu,\ \beta_1,\ \ldots,\ \beta_k;\ \gamma,\ \gamma_{k+1},\ \ldots,\ \gamma_n;\ x_1,\ \ldots,\ x_n],$$

provided that $0<\mathrm{Re}\ (\beta_i)<\mathrm{Re}(\nu_i)$; $0<\mathrm{Re}\ (\mu_i)<\mathrm{Re}\ (\nu_i+\lambda_i-\beta_i)$, where $i=1, \ldots, k$.

[Extension of (2.5)]

$$(3.5)\quad R\left\{F\left[\begin{array}{c|c|c}\begin{matrix}1\\1\\k\\0\\1\\k\end{matrix} & \begin{matrix}\alpha\\(-;\nu)\\\lambda_1, \nu_1; \ldots; \lambda k, \nu k\\\ldots\ \ \ldots\ \ \ldots\\(\gamma;-)\\\mu_1; \ldots; \mu k\end{matrix} & x_1\ t_1, \ldots, x_k t_k, x_{k+1}, \ldots, x_n\end{array}\right]\right\}$$

$$= \underset{(4)\ CD}{\overset{(k)\ (m)}{\Phi}}\ [\alpha, \nu, \beta_1, \ldots, \beta_k; \gamma; x_1, \ldots, x_n]$$

valid if $0<\mathrm{Re}\ (\mu_i)<\mathrm{Re}\ (\nu_i+\lambda_i-\beta_i)$; $0<\mathrm{Re}\ (\beta_i)<\mathrm{Re}\ (\nu_i)$, $i=1, \ldots, k$.

[Extension of (2.6)]

$$(3.6)R\left\{F\left[\begin{array}{c|c|c}\begin{matrix}1\\1\\k\\0\\1\\n-1\end{matrix} & \begin{matrix}\alpha\\(\lambda_1;\nu)\\\gamma_1; \lambda_2, \nu_2; \ldots; \lambda k\ \nu k,\\\ldots\ \ \ldots\ \ \ldots\\(\nu_1, \mu_1; -)\\\mu_2; \ldots; \mu k; \gamma k; \gamma_{k+1}; \ldots; \gamma_n\end{matrix} & \begin{matrix}x_1 t_1, x_1\ x_2 t_1 t_2, \ldots, x_1 x_k\\ t_1 t_k, x_{k+1}, \ldots, x_n\end{matrix}\end{array}\right]\right\}$$

$$= \overset{(k)\ (n)}{F_{CD}}\ [\alpha, \nu, \gamma_1, \beta_1, \ldots, \beta_k; \nu_1, \gamma_{k+1}, \ldots, \gamma_n; x_1, x_1 x_2, \ldots, x_1 x_k, x_{k+1}, \ldots, x_n],$$

provided that $0<\mathrm{Re}\ (\mu_i)<\mathrm{Re}\ (\nu_i+\lambda_i-\beta_i)$; $0<\mathrm{Re}\ (\beta_i)<\mathrm{Re}\ (\nu_i)$, $i=1, \ldots, k$.

[Extension of (2.7)]

$$(3.7)\ R\left\{F\left[\begin{array}{c|c|c}\begin{matrix}1\\1\\k\\0\\1\\n-1\end{matrix} & \begin{matrix}\alpha\\(\lambda_1;-)\\\gamma_1; \nu_2, \lambda_2; \ldots; \nu k, \lambda k\\\ldots\ \ \ldots\ \ \ldots\\(\beta_1, \mu_1; -)\\\mu_2; \ldots; \mu k; \gamma k+1; \ldots; \gamma_n\end{matrix} & \begin{matrix}x_1 t_1, x_1 t_1 x_1 t_2, \ldots, x_1 t_2 x_k t_k,\\ x_{k+1}, \ldots, x_n\end{matrix}\end{array}\right]\right\}$$

$$= \underset{(2)\ CD}{\overset{(k)\ (n)}{\Phi}}\ [\alpha, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1, \gamma_{k+1}, \ldots, \gamma_n; x_1, x_1 x_2, \ldots, x_1 x_k, x_{k+1}, \ldots, x_n],$$

provided that $0<\mathrm{Re}\ (\mu_i)<\mathrm{Re}\ (\nu_i+\lambda_i-\beta_i)$; $0<\mathrm{Re}\ (\beta_i)<\mathrm{Re}\ (\nu_i)$, $i=1, \ldots, k$.

[Extension of (2.8)]

$$(3.8)\ R\left\{F\left[\begin{array}{c|c|c}\begin{matrix}0\\1\\k\\0\\1\\n-1\end{matrix} & \begin{matrix}\ldots\ \ \ldots\ \ \ldots\\(\lambda_1;\nu)\\\gamma_1; \nu_2, \lambda_2; \ldots; \nu k, \lambda k\\\ldots\ \ \ldots\ \ \ldots\\(\beta_1, \mu_1; -)\\\mu_2; \ldots; \mu k; \gamma k+1; \ldots, \gamma_n\end{matrix} & \begin{matrix}x_1 t_1, x_1 t_1 x_2 t_2, \ldots, x_1 t_1 x_k t_k\\ x_{k+1}, \ldots, x_n\end{matrix}\end{array}\right]\right\}$$

$$= {}^{(k)}_{(3)}\Phi^{(n)}_{CD}[v, \gamma_1, \beta_2, \ldots, \beta_k; v_1, \gamma_{k+1}, \ldots, \gamma_n; x_1, x_1x_2, \ldots, x_1x_k, x_{k+1} \ldots, x_n],$$

provided that $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}(v_i)$; $0 < \mathrm{Re}\,(\mu_i) < \mathrm{Re}\,(v_i + \lambda_i - \beta_i)$, $i = 1, \ldots, k$.

[Extension of (2.9)]

$$(3.9)\quad R\left\{F\left[\begin{array}{c|c|c} 1 & \alpha & \\ 1 & (\lambda_1; v) & \\ k & \gamma_1; v_2, \lambda_2; \ldots; v_k, \lambda_k & x_1t_1, x_1t_1x_2t_2, \ldots, x_1t_1x_kt_k, \\ 0 & \ldots \quad \ldots \quad \ldots & x_{k+1}, \ldots, x_n \\ 1 & (\beta_1, \mu_1; -) & \\ k-1 & \mu_2; \ldots; \mu_k & \end{array}\right]\right\}$$

$$= {}^{(k)}_{(4)}\Phi^{(n)}_{CD}[\alpha, v, \gamma_1, \beta_2, \ldots, \beta_k; v_1; x_1, x_1x_2, \ldots, x_1x_k, x_{k+1}, \ldots, x_n],$$

provided that $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}(v_i)$; $0 < \mathrm{Re}(\mu_i) < \mathrm{Re}(v_i + \lambda_i - \beta_i)$, $i = 1, \ldots k$.

[Extension of (2.10)]

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