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Best Approximation and Fixed Point Theorems*

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The purpose of this paper is to use a fixed point theorem of Edelstein [5] in order to obtain a theorem of best approximation. Further fixed point theorems will be inferred.

Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $a \in X$; then there exists a unique point $P(a) \in C$ closest to a, *i.e.*

$$\| P(a) - a \| = \inf_{x \in C} \| x - a \| =: d(a, C)$$

(*e.g.*, see [3, *p.* 22]). The map $P : X \to C$ is called the proximity map. If X is a Hilbert space then P is a nonexpansive map [4, theorem 3].

Proposition 1. If $f : C \to X$ is a map from C into X and u is a fixed point of $P \circ f$, then

$$\| u - f(u) \| = d(f(u), C).$$

Various authors combine this proposition with known fixed point theorems to obtain criteria for the existence of a point u for which $\| u - f(u) \| = d(f(u), C)$; from this best approximation results they get further fixed point theorems ([1], [2], [6], [7], [8], [9]).

In this note, using the same idea, we combine proposition 1 with the fixed point theorem [5, theorem 2] of of Edelstein to obtain a criterion for the existence of a point $u \in C$ of minimal distance from $f(u)$ for a given map $f : C \to X$ (theorem 1). From this result a fixed point theorem (theorem 3) will be deduced. Our results generalize theorems by Singh and Watson [9, theorem 5 and theorem 6] and by Schoneberg [9, theorem 6].

First we recall the notion of the asymptotic center of a sequence $\{u_n\}$, which is essential in [5]. Let $\{u_n\}$ be a bounded sequence in C. Define

$$r_m(y) := \sup \{ \| u_k - y \| : k \geqslant m \} \ (y \in X).$$

It is known that there exists a unique point $c_m \in C$ such that

$$r_m(c_m) = \inf \{ r_m (y \in C \}.$$

Edelstein [5, theorem 1] proves that $\{c_m\}$ converges to a point $u \in C$ called the asymptotic center of $\{u_n\}$ with respect to C. The following result due to Edelstein will be used in our theorem.

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Theorem 1. (Edelstein [5, theorem 2]). Let $f: C \to C$ be a map from C into C and let $x \in C$ be such that the sequence defined by $u_n = f^n(x)$ is bounded. Denote by u the asymptotic center of $\{u_n\}$ with respect to C. Suppose that there exists a positive integer n_0 and neighborhood V of u in C such that

$$\| f^k(x) - f(v) \| \leqslant \| f^{k-1}(x) - v \| \quad (k \geqslant n_0,\ v \in V).$$

Then $$f(u) = u.$$

Now from proposition 1 and theorem 1 it follows immediately :

Theorem 2. Let $f: C \to X$ be a map from C into X and let $x \in C$ be such that the eequence defined by $u_n := (P \circ f)^n (x)$ is bounded. Denote by u the asymptotic center of $\{u_n\}$ with respect to C. Suppose that there exists a positive integer n_0 and a neighbourhood V of u such that

$$\| (P \circ f)^k (x) - (P \circ f)(v) \| \leqslant \| (P \circ f)^{k-1} (x) - v \|$$

for all integers $k \geqslant n_0$ and all $v \in V \cap C$. Then u is a fixed point of $P \circ f$ and $\| u - f(u) \| = d(f(u), C)$.

Theorem 3. Let X be a Hilbert space and $f: C \to X$ a continuous map such that $f(C)$ is bounded. Suppose that for every $x \in C$ there is a positive integer $n = n(x)$ such that

$$\| (P \circ f)^k (x) - (P \circ f)^k (y) \| \leqslant \| (P \circ f)^{k-1} (x) - (P \circ f)^{k-1} (y) \|$$

for all integers $k \geqslant n$ and all $y \in C$. Then there exists a fixed point u of $P \circ f$ and $\| u - f(u) \| = d(f(u), C)$.

Proof. Since the map P is nonexpansive, the closed convex hull $B := \overline{co}\,(P \circ f)(C)$ is bounded and $P \circ f$ is continuous. Therefore, by [5, corollary] the map $P \circ f_{|B} : B \to B$ has a fixed point $u \in B$. Hence one obtaine as in proposition 1

$$\| u - f(u) \| = \| P(f(u) -) f(u) \| = d(f(u), C).$$

If f is nonexpansive and X is a Hilbert space, then $(P \circ f)^k$ is nonexpansive for every positive integer k. Therefore, from theorem 3, one gets the following of Singh and Watson.

Corollary 1. (Singh and Watson [9, theorem 5]). Let X be a Hilbert space and $f: C \to X$ be a nonexpansive map such that $f(C)$ is bounded. Then there exists a point $u \in C$ such that

$$\| u - f(u) \| = d(f(u), C).$$

Remark. It can happen that $P \circ f$ is nonexpansive while f is not : take, *e.g.* $f: [0, 1] \to R$, $f(x) = 2x + 1$. Here the assumptions of theorems 1 and 2 are satisfied, but not those of corollary 1.

Until now we have studied conditions for the existence of a point $u \in C$ with $\| u - f(u) \| = d(f(u), C)$. We will now study how to get fixed point theorems from such results. The following lemma formalizes the method just used in the proof of [9, theorm 6].

Lemma. Let X be a Hilbert space and $f : C \to X$ a map; let $u, y \in C$ be such that u is a fixed point of $P \circ f$ and $\| f(u)-y \| \leqslant \| u-y \|$, Then u is a fixed point of f.

Proof. The segment $S := \{y+t(u-y) : t \in [0, 1]\}$ is contained in C and from $\| u-f(u) \| = \| (P \circ f)(u)-f(u) \| = d(f(u), C)$, it follows that $d(f(u), S) \leqslant \| u-f(u) \| = d(f(u), C) \leqslant d(f(u), S)$, *i. e.*

$$\| u-f(u \| = d((f(u), S). \quad ...(1)$$

The segment S joins the center y of B and the point u of the boundary ∂B. But, since X is a Hilbert space, for every point $z \in B/\{u\}$ we have $\| z-u \| \neq d(z,S)$, and so thanks to (1), we conclude $u = f(u)$.

Theorem 4. Let X be a Hilbert space and suppose that the assumptions either in theorem 2 or in theorem 3 or in corollary 1 are satisfied. Assume further that there is an element $y \in C$ such that $\| f(x)-y \| \leqslant \| x-y \|$ for every $x \in \partial C$. Then f has a fixed point.

Proof. Our hypotheses imply that the map $P \circ f$ has a fixed point $u \in C$ and $\| u-f(u) \| = d(f(u), C)$. Suppose that $f(u) \notin C$. Then $u \in \partial C$, hence $\| f(u)-y \| \leqslant \| u-y \|$ and, by the lemma, $f(u) = u \in C$. If $f(u) \in C$, then $f(u) = (P \circ f)(u) = u$.

Theorem 4 contains the following result by Schöneberg.

Corollary 2. (Singh and Watson [9, theorem 6]). Let X be Hilbert space and $f : C \to X$ a nonexpansive map such that $f(C)$ is bounded. Assume further that there is an element $y \in C$ such that $\| f(x)-y \| \leqslant \| x-y \|$ for every $x \in \partial C$. Then f has a fixed point.

Proposition 2. Let B_r $(r > 0)$ be the closed ball in X with center at the origin and radius r. Let $f : B_r \to X$ be a map with the property that

"if $f(x) = ax$ for some x on the boundary of B_r then $a \leqslant 1$."

Then, if u is a fixed point of $P \circ f$, u is also a fixed point of f.

Proof. By proposition 1 we have $\| u-f(u) \| = d(f(u), B_r)$. Therefore it is enough to prove that $f(u) \in B_r$. Suppose that $f(u) \notin B_r$. Then u is on the boundary of B_r and $f(u) = ax$ for some $a > 0$. By assumption $a \leqslant 1$ and therefore $f(u) = au \in B_r$.

From proposition 2 one can obtain fixed point theormes under conditions which imply a fixed point for $P \circ f$, *e.g.* under the assumption either theorem 2 or theorem 3 or corollary 1. If one assumes as in corollary 1 that X is a Hilbert space and f is a nonexpansive map, then one obtains from proposition 2 the theorem by Singh and Watson [9, theorem 8].

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