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Petryshyn's Condition For Ky Fan Type Theorem*

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0. INTRODUCTION

The following well-known theorem due to Ky Fan [3, Theorem 8] has applications in fixed point theory, approximation theory, minimax theory and avriational inequalities.

Let C be a nonempty compact convex subset of a normed linear space X and $f : C \to X$ a continuous function. Then there is a point $u \in C$ such that

$$\| u - f(u) \| = d(f(u), C) := \inf \{ \| x - f(u) \| : x \in C \}.$$

Lin [5, Theorem 2] proved a theorem of this type for a continuous set-condensing map defined on a nonempty closed bounded convex subset of a Hilbert space. A result analogous to that of Lin [5] was obtained by Singh and Watson [10] for nonexpansive maps.

Lin and Yen [6, Theorem 1] gave a common generalization of these results above replacing continuous set-condensing maps or nonexpansive maps, respectively, by continuous 1-set-contractive maps for which a certain range is closed. Obviously any set-condensing or non-expansive map is a 1-set-contractive map; further the closedness condition of [6, Theorem 1] is satisfied for a set-condensing or nonexpansive map f (defined on a closed convex bounded subset C of a Hilbert space) since in that case $(I - f)(C)$ is closed.

The condition that $(I-f)(C)$ is closed also appears in Browder's fixed point theorem [1, *p*. 230] to generalize the fixed point theorems given in ([4] or [9]) and [1, *p*. 104] for continuous set-condensing and nonexpansive maps, respectively.

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Petryshyn [8, Theorem 1] showed that in Browder's fixed point theorem for a map $f: C \to X$ the condition that $(I-f)(C)$ is closed can be replaced by the weaker condition

$$0 \in \overline{(I-f)(C)} \Rightarrow 0 \in (I-f)(C),$$

or equivalently, by

(P) if $\{x_n\}$ is any sequence in C such that $x_n - f(x_n) \to 0$ $(n \to \infty)$, then there exists $x \in C$ with $x = f(x)$.

In this note we show that several other theorems can be improved replacing the closedness condition by the condition (P) of Petryshyn, in particular [6, Theorem 1] of Lin and Yen mentioned above.

1. KY FAN TYPE THEOREM

We use the terminology of Petryshyn [8]. In the following let X be a Banach space and C be a nonempty closed convex subset of X.

Lemma 1. *Assume that $g: C \to C$ is a continuous 1-set-contractive map and $f(C)$ is bounded. Then there is a sequence $\{x_n\}$ in C such that $x_n - g(x_n) \to 0$ $(n \to \infty)$.*

Proof. Let $B := \overline{co}g(C)$ the closed convex hull of $g(C)$ and $x_n \in B$. Put $k_n = 1 - \frac{1}{n}$ and $g_n(x) = k_n g(x) + (1-k_n)x_0$ for $x \in B$ and for all positive integers n. Then $g_n : B \to B$ is a continuous k_n-set-contractive map and therefore has by [4, Theorem 3] a fixed point $x_n \in B$. Since

$$x_n - g(x_n) = \frac{1}{n}\, g(x_n) - \frac{1}{n} x_0$$

and $\{g(x_n)\}$ is bounded, the sequence $\{x_n - g(x_n)\}$ converges to 0.

If in addition to the assumption of Lemma 1, the map g satisfies the following condition (P) of the introduction :

(P) If $\{x_n\}$ is any sequence in C such that $x_n - g(x_n) \to 0$ $(n \to \infty)$, then there exists $x \in C$ with $x = g(x)$,

then g has a fixed point by Lemma 1.

The following theorem generalizes [6, Theorem 1].

Theorem 1. *Let X be a Hilbert space, $f: C \to X$ be a continuous 1-set-contractive map such that $f(C)$ is bounded and assume that $p \circ f$ satisfies the condition (P), where p is the proximity map for C (in the sense of [2]). Then there exists a point u in C such that*

$$\| u - f(u) \| = d(f(u), C).$$

Proof. By [2, Theorem 3], p is a nonexpansive map. Therefore $p \circ f : C \to C$ is a continuous 1-set-contractive map and $(p \circ f)(C)$ is bounded, since $f(C)$ is bounded.

It follows by Lemma 1 that $p \circ f$ has a fixed point $u \in C$. Hence

$$\| u-f(u) \| = \| p(f(u)) - f\ u) \| = d(f(u), C).$$

The role, which the proximity map plays in Theorem 1, is played in Theorem 2 by the radial retraction : Let $r>0$ and B_r be the closed ball with center at the origin and radius r. Define $R : X \to B_r$ by

$$R(x)=\begin{cases} x & \text{if } \| x \| \leqslant r \\ \dfrac{rx}{\| x \|} & \text{if } \| x \| \geqslant r . \end{cases}$$

Then $\| R(x)-x \| = d(x, B_r)$.

The following theorem generalizes [5, Theorem 1].

Theorem 2. *Assume that $f : B_r \to X$ is a continuous 1-set-contractive map and that $R \circ f$ satisfies (P). Then there exists a point u in B_r such that*

$$\| u-f(u) \| = d(f(u), B_r).$$

Proof. By [7, Proposition 9] R is a continuous 1-set-contractive map, therefore $R \circ f : B_r \to B_r$ is a continuous 1-set-contractive map. It follows from Lemma 1 that $R \circ f$ has a fixed point $u \in B_r$. Hence

$$\| u - f(u) \| = \| R(f(u)) - f(u) \| = d(f(u), C).$$

Lin showed in [6, Section 3] that one can obtain various fixed point theorems with the help of his best-approximation theorem [6, Theorem 1]. In these fixed point theorems summarized in [6, Theorem 5], there is always assumed that $(I-p \circ f\,(C)$ is closed.

With the help of our generalization of [6, Theorem 1 of this paper, we can weaken the assumption above to the assumption that $p \circ f$ satisfies the condition (P).

We now give an example of a 1-set-contractive map f, such that f has a fixed point, hence satisfies (P), but the range of $I-f$ is not closed.

Example. Let B be the closed unit ball with center at the origin in the space l_2. Define $T : B \to l_2$ by

$$T(x) = (\| x \|_2, x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \ldots), \ x=(x_n) \in B.$$

Then $f : = I-T$ is a 1-set-contractive map, $f(0)=0$ and $e_1=(1, 0, 0, \ldots)$ $\in \overline{(I-f)(B)} / (I-f)(B)$.

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