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Associated Operational Calculus For a Bessel-Clifford Operator

By

J. Rodriguez

Facultad de Matematicas, Universidad de La Laguna, La Laguna, Tenerife, Canary Islands, Spain

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ABSTRACT

In this paper an operational calculus for a certain Bessel-Clifford operator is developed by following an algebraic process similar to Mikusinski's. To achieve this a new convolution is introduced, which proves an extension to Koh's. Two applications derived from the ensuing calculus are shown.

1. INTRODUCTION

Ditkin [2], followed further by Ditkin and Prudnikov [3], has developed and operational calculus for the DtD-operator, through an algebraic process similar to Mikusinski's.

This calculus has been extended by Meller [10] to the operator $B_\nu = t^{-\nu} D t^{\nu+1} D$, when $-1<\nu<1$, and recently Koh [6], on using the Riemann-Liouville fractional derivative, extends this operational calculus (for the B_ν-operator) to the interval $-1<\nu<+\infty$. The results obtained in [6] include those given in [3] for $\nu=0$ and in [10] for $\nu \in (-1, 1)$.

In this paper an operational calculus for the operator $A_{\alpha,\beta} = t^{1-\alpha-\beta} Dt^{\alpha} Dt^{\beta}$, $\alpha \geqslant 1$ and $\beta \geqslant 0$, has been developed, which generalizes the previous work of Koh when $\alpha=\nu+1$ and $\beta=0$ [6], and the results of Ditkin [2] and Ditkin-Prudnikov [3], for $\alpha=1$ and $\beta=0$, as well.

2. THE RIEMANN-LIOUVILLE FRACTIONAL INTEGRATION AND DIFFERENTIATION

The Riemann-Liouville integral or order $\nu>0$, is defined as (see [14]) :

$$I^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} f(t)dt \qquad \ldots(2.1)$$

Some well-known properties of this operator are as follows :

$$I^0 f(t) = \lim_{\nu \to 0} I^\nu f(t) = f(t) \qquad \ldots(2.2)$$

$$DI^{\nu+1} f(t) = I^\nu f(t), \ \nu > 0 \qquad \ldots(2.3)$$

$$I^\nu I^\mu f(t) = I^{\nu+\mu} f(t), \ \nu, \mu > 0 \qquad \ldots(2.4)$$

$$I^\nu t^k = \frac{\Gamma(k+1)}{\Gamma(\nu+k+1)} t^{\nu+k} \ (k+1>0, \ \nu \geqslant 0) \qquad \ldots(2.5)$$

whereas the Riemann-Liouville fractional derivative of order $\nu>0$ of a function $f(t) \epsilon C^n$ ((0, ∞)) is defined to be $D^\nu f(t) = D^n I^{n-\nu} f(t)$ $(n-1<\nu \leqslant n)$.

From (2.1) the following properties can be derived :

$$D^\nu I^\mu f(t) = I^{\mu-\nu} f(t) \quad (\nu, \mu, \mu-\nu \epsilon K^* \cup \{0\}) \qquad \ldots(2.6)$$

$$D^\nu t^k = \frac{\Gamma(k+1)}{\Gamma(k-\nu+1)} t^{k-\nu} \quad (-1<k<\infty, \ 0<\nu \leqslant k) \qquad \ldots(2.7)$$

$$D^\nu f(t) = I^{-\nu} f(t). \qquad \ldots(2.8)$$

3. THE EXTENSION OF C^∞ TO THE QUOTIENT FIELD

Let α and β be real numbers with $\alpha \geqslant 1$ and $\beta \geqslant 0$. Consider the set of functions denoted as C^∞ ([0, ∞)), where the following operation is defined :

$$f(t)*g(t) = \frac{1}{\Gamma(\alpha+\beta)\Gamma(\beta+1)} t^{-\beta} D^{\beta+1} t^{2-\alpha+\beta} D^{\alpha+\beta}.$$

$$\cdot \int_0^t (t-\xi)^{\alpha+\beta-1} \xi^{\alpha+\beta-1} d\xi \int_0^1 \eta^\beta (1-\eta)^\beta f(\xi\eta) g[(1-\eta)(t-\xi)] d\eta. \qquad \ldots(3.1)$$

for any pair $f(t), g(t) \epsilon C^\infty$.

From the very definitions for the fractional derivative and integral it can be easily seen that :

$$t^p * t^q = \frac{\Gamma(\beta+p+1)\,\Gamma(\beta+q+1)\,\Gamma(\alpha+\beta+p)\,\Gamma(\alpha+\beta+q)}{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)\,\Gamma(\alpha+\beta+p+q)\,\Gamma(\beta+p+q+1)} t^{p+q} \quad (3.2)$$

holds for $p, q \epsilon N \cup \{0\}$.

Hence, by invoking Weierstrass's approximation theorem (3.2) can be proved for any pair $f(t)$, $g(t)$ of functions in C^∞ . Then, the $*$-operation is closed on C^∞ . Moreover, if $f(t)$, $g(t)$, $h(t)$ belong to C^∞ , then the following list of properties holds :

(*i*) $f(t)*g(t) = g(t)*f(t)$
(*ii*) $f(t)*(g(t)*h(t)) = (f(t)*g(t))*h(t)$
(*iii*) $f(t)*(g(t)+h(t)) = f(t)*g(t) + f(t)*h(t)$
(*iv*) $A_{\alpha,\beta}(f(t)*g(t)) = (A_{\alpha,\beta} f(t))*g(t) = f(t)*(A_{\alpha,\beta}\, g(t))$

Note 1. Since (*iv*) holds, we shall say according to Dimovski [1], that the sign$*$ stands for a convolution for the operator $A_{\alpha,\beta}$ on C^∞ .

Proposition 1. For any complex number λ and any $f(t)\epsilon C^{\infty}$, we have

$$\lambda^{*}f(t)=\lambda f(t).$$

Proof. From the previous, the following can be inferred.

$$\lambda^{*}f(t)=\frac{\lambda}{\Gamma(\alpha+\beta)\Gamma(\beta+1)}t^{-\beta}D^{\beta+1}t^{2-\alpha+\beta}D^{\alpha+\beta}.$$

$$\int_0^t(t-\xi)^{\alpha+\beta-1}\xi^{\alpha+\beta-1}d\xi\int_0^1\eta^{\beta}(1-\eta)^{\beta}f(\xi\eta)d\eta$$

$$=\lambda t^{-\beta}D^{\beta+1}t^{2-\alpha+\beta}D^{\alpha+\beta}I^{\alpha+\beta}t^{\alpha-\beta-2}I^{\beta+1}t^{\beta}f(t)=\lambda f(t).$$

Proposition 2. If $f(t)^{*}g(t)=0$, $f(t)$, $g(t)\epsilon C^{\infty}$, then either of these functions equals zero, i. e. C^{∞} has no zero divisors.

Proof. Set $f(t)^{*}g(t)=0$. Since $t^{-\beta}\neq 0$ $(0,\infty)$, then the following can be established

$$D^{n+1}I^{n-\beta}t^{2-\alpha+\beta}D^{\alpha+\beta}\int_0^t(t-\xi)^{\alpha+\beta-1}\xi^{\alpha+\beta-1}.$$

$$\int_0^1\eta^{\beta}(1-\eta)^{\beta}f(\xi\eta)g[(1-\eta)(t-\xi)]d\eta d\xi=0.$$

Hence

$$\int_0^t(t-\xi)^{n-\beta-1}\xi^{2-\alpha+\beta}D^{\alpha+\beta}\int_0^{\xi}(\xi-\eta)^{\alpha+\beta-1}\eta^{\alpha+\beta-1}.$$

$$\cdot\int_0^1 x^{\beta}(1-x)^{\beta}f(\eta x)g[(1-x)(\xi-\eta)]dx d\eta d\xi=$$

$$=C_1\frac{t^n}{n!}+C_2\frac{t^{n-1}}{(n-1)!}+\dots+C_n t+C_{n+1}\qquad\dots(3.3)$$

is true.

As a consequence, $C_{n+1}=0$ for $t=0$. But it is well-known that if $C_1\neq 0$ for some $i\epsilon(1,2,\dots n)$, then $f(t)$ and $g(t)$ are polynomials and in this case it follows from Weierstrass's approximation theorem that the left side of (3.3) is of a degree greater than n. Thus, the right side of (3.3) must be zero.

Now, by following a similar line of reasoning and by invoking further Titchmarsh's theorem, we can state that

$$\int_0^t\int_0^1(t-\xi)^{\alpha+\beta-1}\xi^{\alpha+\beta-1}x^{\beta}(1-x)^{\beta}f(\xi x)g[(1-x)(t-\xi)]dx\,d\xi=0\quad(3.4)$$

holds.

Now, if we perform in (3.4) the following change of variables $x=\frac{y}{\tau}$, $\xi=z\tau$, $t=v\tau$, we obtain

$$\int_0^v \int_0^\tau \tau^{\alpha-2}\ z^{\alpha+\beta-1}\ y^\beta\ f(zy)\ (\tau-y)^\beta\ (v-z)^{\alpha+\beta-1}$$

$$g[(\tau-y)\ (v-z)]\ dy\ dz=0. \qquad ...(3.5)$$

and then from Mikusinski's theorem and Ryll's-Nardzewski's as well, [12], it can be finally inferred that either of the following equations holds :

$$z^{\alpha+\beta-1}\ y^\beta f(zy)=0 \text{ or } (\tau-y)^\beta\ (v-z)^{\alpha+\beta-1} g[(\tau-y)(v-z)=0$$

Hence, either $f(t)$ is $g(t)$ is zero, which completes the proof.

By virtue of proposition 1, it follows that the unit element for * in C^∞ is the plain number 1.

We can therefore establish the following :

Proposition 3. The set C^∞ endowed with the ordinary sum and * as multiplication, becomes a unitary commutative ring without zero divisors, which can be extended to a quotient field

$$M=C^\infty \times (C^\infty - \{0\})/\sim,$$

where equivalence $\sim$ is defined on $C^\infty \times (C^\infty - \{0\})$ in the usual form, say

$$(f(t), g(t))\sim(\bar{f}(t), \bar{g}(t)) \Leftrightarrow f(t)*\bar{g}(t)=g(t)*\bar{f}(t).$$

From now on, the pair $(f(t), g(t))$ will be noted as $\frac{f(t)}{g(t)}$ and will be regarded as an operator.

Define on M the usual operations of addition, multiplication and multiplication by scalar, as :

$$\frac{f(t)}{g(t)}+\frac{\bar{f}(t)}{\bar{g}(t)}=\frac{f(t)*\bar{g}(t)+\bar{f}(t)*g(t)}{g(t)*\bar{g}(t)}$$

$$\frac{f(t)}{g(t)}\cdot\frac{\bar{f}(t)}{\bar{g}(t)}=\frac{f(t)*\bar{f}(t)}{g(t)*\bar{g}(t)}$$

$$\lambda\cdot\frac{f(t)}{g(t)}=\frac{\lambda f(t)}{g(t)}$$

Thus, M becomes an algebra.

The quotient set M contains a subset M', which is isomorphic to C^∞, th ough the following application

$$M'\subset M \to C^\infty$$

$$\frac{f(t)}{1} \to f(t).$$

Hence, operators of the form $\frac{f(t)}{1}$ constitute a subring of M.

4. AN OPERATIONAL CALCULUS

Let us now prove that the operator $A_{\alpha,\beta}$ belongs to M. First of all, let us state the following,

Proposition 4. The operator $A_{\alpha,\beta}$ can be expressed as :

$$A_{\alpha,\beta} = t^{1-\alpha-\beta} D t^{\alpha} D t^{\beta} = t^{-\beta} D t^{2-\alpha} D t^{\alpha+\beta-1} \qquad \ldots(4.1)$$

Proof. Both operators reduce to the differential operator,

$$tD^2 + (\alpha+2\beta)D + t^{-1}\beta(\alpha+\beta-1).$$

On the other hand the right hand inverse operator for $A_{\alpha,\beta}$ is given as:

$$R_{\alpha,\beta}\, f(t) = t^{-\beta} \int_0^t \xi^{-\alpha} d\xi \int_0^{\xi} \eta^{\alpha+\beta-1} f(\eta) d\eta = t^{1-\alpha-\beta} \int_0^t \xi^{\alpha-2} d\xi \int_0^{\xi} \eta^{\beta} f(\eta) d\eta.$$

due to the fact that $A_{\alpha,\beta} \cdot R_{\alpha,\beta}\, f(t) = f(t)$ is true for each $f(t) \in C^{\infty}$.

Now, if we restrain the domain of the operator $A_{\alpha,\beta}$ to be the set.

$$\{ f(t) / f(t) \in C^{\infty} ,\ t^{\beta} f(t) \cdot |_{t=0^+} = 0 \},$$

then $R_{\alpha,\beta}$ becomes a left-hand inverse operator for $A_{\alpha,\beta}$. That is to say,

$$R_{\alpha,\beta}\, A_{\alpha,\beta}\, f(t) = f(t).$$

Proposition 5. Equality

$$\frac{t}{(\alpha+\beta)(\beta+1)} * f(t) = R_{\alpha,\beta}\, f(t) \text{ holds for any } f(t) \in C^{\infty} .$$

Proof. Starting from

$$\frac{t}{(\alpha+\beta)(\beta+1)} * f(t) = \frac{1}{\Gamma(\alpha+\beta+1)\,\Gamma(\beta+2)}\, t^{-\beta} D^{\beta+1} t^{2-\alpha+\beta} D^{\alpha+\beta}$$

$$\int_0^t (t-\xi)^{\alpha+\beta} \xi^{a+\beta-1} d\xi \cdot \int_0^1 \eta^{\beta}(1-\eta)^{\beta+1} f(\xi\eta)\, d\eta.$$

and by performing the change of variable $\xi\eta = x$, we obtain :

$$t^{-\beta} D^{\beta+1} t^{2-\alpha+\beta} D^{\alpha+\beta} I^{\alpha+\beta+1} t^{\alpha-\beta-3} I^{\beta+} t^{\beta} f(t) =$$

$$= t^{-\beta} I I^{-\beta-2} t^{2-\alpha+\beta} I^{\alpha} I^{1-\alpha} t^{\alpha-\beta-3} I^{\beta+2} t^{\beta} f(t). \qquad \ldots(4.2)$$

But since $-\beta-2+2-\alpha+\beta+\alpha=0$, it can be stated that (see [9])

$$I^{-\beta-2} t^{2-\alpha+\beta} I^{\alpha} = t^{-\alpha} I^{\alpha-\beta-2} t^{\beta+2}.$$

As a consequence, (4.2) can be written as

$$t^{-\beta} I t^{-\alpha} I^{\alpha-\beta-2} t^{\beta+2} I^{1-\alpha-\beta-3} I^{\beta+2} t^{\beta} f(t) =$$

$$= t^{-\beta} I t^{-\alpha} I I^{\alpha-\beta-3} t^{\beta+2} I^{1-\alpha} t^{\alpha}\ {}^{\beta-3} I^{\beta+2} t^{\beta} f(t). \qquad \ldots(4.3)$$

And now by applying once more the second index law [9] to

$$I^{\alpha-\beta-3} t^{\beta+2} I^{1-\alpha} = t^{\alpha-\beta-2} t^{3+\beta-\alpha},$$

then (4.) becomes

$$t^{-\beta}\, It^{-\alpha}\, It^{\alpha-1} I^{-\beta-2} t^{3+\beta-2}\, t^{\alpha-\beta-3}\, I^{\beta+2} t^{\beta}\, f(t)$$

$$=t^{-\beta} It^{-\alpha}\, It^{\alpha+\beta-1} f(t)=t^{-\beta}\int_0^t \xi^{-\alpha}\, d\xi \int_0^{\xi} \eta^{\alpha+\beta-1} f(\eta)\, d\eta.$$

From this result the following can be established by induction.

Prodosition 6. Letting $f(t)\epsilon C^{\infty}$ and $k\epsilon N$, then the following holds.

$$\frac{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+k)\,\Gamma(\beta+k+1)}\, t^{k*} f(t)=R^{k}_{\alpha,\,\beta} f(t).$$

Proof. The previous expression has already been verified for the case $k=1$.

Suppose now that it is valid for $k=m$ and let us see that it also holds for $k=m+1$

$$\frac{t}{(\alpha+\beta)\,(\beta+1)} * \left(\frac{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+m)\,\Gamma(\beta+m+1)}\, t^{m*} f(t)\right)$$
$$=R_{\alpha,\beta}\left(R^{m}_{\alpha,\,\beta}\, f(t)\right)$$

$$\Rightarrow\left(\frac{t}{(\alpha+\beta)\,(\beta+1)} * \frac{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+m)\,\Gamma(\beta+m+1)}\, t^{m}\right) * f(t)$$
$$=R^{m+1}_{\alpha,\,\beta}\, f(t)$$

which reduces to

$$\frac{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+m+1)\,\Gamma(\beta+m+2)}\, t^{m+1} * f(t)=R^{m+1}_{\alpha,\,\beta} f(t).$$

Hence, the operators $R^{k}_{\alpha,\,\beta}$ belong to M and can be therefore identified with the following expression :

$$\frac{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+k)\,\Gamma(\beta+k+1)}\, t^{k}.$$

Besides, the following can be established :

Proposition 7. Equality

$$f(t)=R_{\alpha,\,\beta}\, A_{\alpha,\,\beta} f(t)+t^{-\beta}[t^{\beta} f(t)]\big|_{t=0^+}. \qquad \ldots(4.4)$$

holds for any $f(t)\epsilon C^2([0,\,\infty))$

Proof. Suffice it to consider that

$$R_{\alpha,\beta}\, R_{\alpha,\beta}\, f(t)=t^{-\beta}\int_0^t \xi^{-\alpha}\, d\xi\int_0^{\xi} \eta^{\alpha+\beta-1} (A_{\alpha,\,\beta} f(\eta)\, d\eta$$
$$=t^{-\beta}\int_0^t \xi^{-\alpha} d\xi\int_0^{\xi} D\eta^{\alpha}\, D\eta^{\beta} f(\eta)\, d\eta=t^{-\beta}\int_0^t D\xi^{\beta} f(\xi)$$
$$=f(t)-t^{-\beta}\,[t^{\beta} f(t)]\big|_{t=0^+}$$ is true.

Now if we set

$$V = \frac{(\alpha+\beta)(\beta+1)}{t}, \qquad \ldots(4.5)$$

then the following can be verified

Proposition 8.

$$Vf(t) = A_{\alpha,\beta} f(t) + t^{-\beta} [t^{\beta} f(t)]_{|t=0^+} V. \qquad \ldots(4.6)$$

holds for any $f(t) \in C^2([0, \infty))$

Proof. It is known that

$$f(t) = R_{\alpha,\beta} A_{\alpha,\beta} f(t) + t^{-\beta} [t^{\beta} f(t)]_{|t=0^+}$$

$$= \frac{t}{(\alpha+\beta)(\beta+1)} * A_{\alpha,\beta} f(t) + t^{-\beta} [t^{\beta} f(t)]_{|t=0^+}$$

then

$$Vf(t) = \left(\frac{(\alpha+\beta)(\beta+1)}{t} * \frac{t}{(\alpha+\beta)(\beta+1)} \right) * A_{\alpha,\beta} f(t) + t^{-\beta} [t^{\beta} f(t)]_{|t=0^+} V$$

$$= A_{\alpha,\beta} f(t) = t^{-\beta} [t^{\beta} f(t)]_{|t=0^+} V.$$

5. OPERATIONAL RULES

Consider the two functions

$$(i) \quad C_{\alpha-1,\beta}(t) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(\alpha+\beta)\Gamma(\beta+1)}{\Gamma(\beta+r+1)\Gamma(\alpha+\beta+r)} t^r \qquad \ldots(5.1)$$

$$(ii) \quad E_{\alpha-1,\beta}(t) = \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+\beta)\Gamma(\beta+1)}{\Gamma(\beta+r+1)\Gamma(\alpha+\beta+r)} t^r \qquad \ldots(5.2)$$

we can formally establish that in $(0, \infty)$

$$t^{-\beta} * C_{\alpha-1,\beta}(at) = \Gamma(\alpha) t^{-\beta} C_{\alpha-1}(at) \qquad \ldots(5.3)$$

$$t^{-\beta} * E_{\alpha-1,\beta}(at) = \Gamma(\alpha) t^{-\beta} E_{\alpha-1}(at) \qquad \ldots(5.4)$$

where $C_{\alpha-1}(at)$ and $E_{\alpha-1}(at)$ are respectively the Bessel-Clifford function of first kind and the modified of first kind, both of them of order $\alpha-1$ [5].

In the following some operational rules are obtained, which will come in useful to certain applications.

As it can be easily proved, the differential equations

$$(A_{\alpha,\,\beta} \pm a)y = 0 \quad (a > 0) \qquad \text{...(5.5)}$$

have the following functions as solution :

$$y_1(t) = 2^{1-\alpha}\, t^{-\beta}\, C_{\alpha_{-1}}(at) \qquad \text{(for the plus-sign)}$$

$$y_2(t) = 2^{1-\alpha}\, t^{-\beta}\, E_{\alpha_{-1}}(at) \qquad \text{(for the minus-sign).}$$

Since

$$\lim_{t \to 0+} t^{\beta}\, y_1(t) = \lim_{t \to 0+} t^{\beta}\, y_2(t) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} \quad \text{is true,}$$

then from 4.6) and (5.5) it can be inferred that

$$V^* f(t) = \mp a^* f(t) + t^{-\beta}\, [t^{\beta} f(t)]_{|t=0^+} V$$

$$\Rightarrow \begin{cases} (V+a)^*(t^{-\beta} * C_{\alpha-1,\,\beta}(at)) = t^{-\beta} * V \\ (V-a)^*(t^{-\beta} * E_{\alpha-1,\,\beta}(at)) = t^{-\beta} * V \end{cases}$$

which leads, among others, to the following operational rules :

(a) $$\frac{V}{V+a} = C_{\alpha-1,\,\beta}(at)$$

(b) $$\frac{V}{V-a} = E_{\alpha-1,\,\beta}(at)$$

(c) $$\frac{aV}{V^2-a^2} = \tfrac{1}{2}\,(E_{\alpha-1,\,\beta}(at) - C_{\alpha-1,\,\beta}(at))$$

(d) $$\frac{V^2}{V^2-a^2} = \tfrac{1}{2}\,(E_{\alpha-1,\,\beta}(at) + C_{\alpha-1,\,\beta}(at))$$

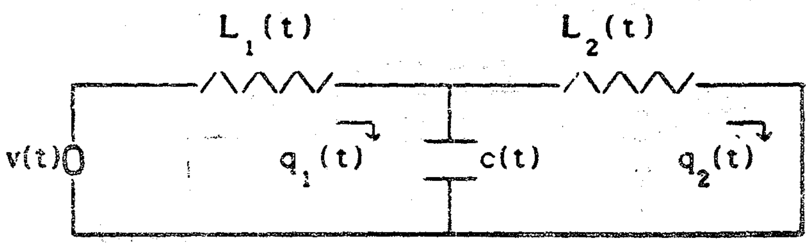
(e) $$\frac{a}{a-V} = 1 - E_{\alpha-1,\,\beta}(at)$$

(f) $$\frac{a}{a+V} = 1 - C_{\alpha-1,\,\beta}(a)$$

6. PRACTICAL APPLICATIONS

(I) Consider the electric network shown in [7], which is reproduced on next page.

This circuit is made up a time-variant source $v(t)$, two inductors *i.e.*, $L_1(t)$ and $L_2(t)$ and a capacitor $c(t)$. An adequate analysis yields the following system of equations :



(Fig. 1)

$$v(t)=DL_1(t)Dq_1(t)+\frac{q_1(t)-q_2(t)}{c(t)}$$
$$0=\frac{q_2(t)-q_1(t)}{c(t)}+DL_2(t)Dq_2(t).$$

If we assume that $L_1(t)=at^{\alpha}$ $L_2(t)=bt^{\alpha}$ and $c(t)=\frac{1}{c}t^{1-\alpha}$, with $a, b, c>0$ and set $q_i(t)=t^{\beta}\ \mu_i(t)$, $i=1, 2$, and $e(t)=t^{1-\alpha-\beta}\ v(t)$, it follows that

$$e(t)=aA_{\alpha,\beta}\ \mu_1(t)+c(\mu_1(t)-\mu_2(t))$$
$$0=c(\mu_2(t)-\mu_1(t))+bA_{\alpha,\beta}\mu_2(t).$$

Now, by invoking (4.6) and assuming $\mu_i(0)=0$, $i=1, 2$, the following can be obtained

$$e(t)=(aV+c)-c\mu_2(t)$$
$$0=c\mu_1(t)+(bV+c)\mu_2(t)$$

therefore,

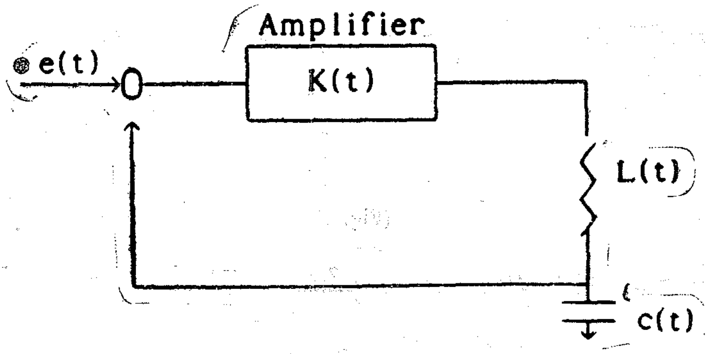
$$\mu_1(t)=\frac{bV+c}{abV^2+c(a+b)V}\cdot e(t)$$

$$\mu_2(t)=\frac{c}{abV^2+c(a+b)V}.e(t).$$

Finally, the previous operational rules yield :

$$\mu_1(t)=\frac{1}{a+b}\left[\frac{b^2}{c(a+b)}-\frac{b^2}{c(a+b)}\cdot C_{\alpha-1,\beta}\left(\frac{c(a+b)}{ab}t\right)\right.$$
$$\left.+\frac{t}{(\alpha+\beta)(\beta+1)}\right]*e(t)$$
$$\mu_2(t)=\left[\frac{ab}{c(a+b)}+\frac{ab}{c(a+b)}\cdot C_{\alpha-1,\beta}\left(\frac{c(a+b)}{ab}t\right)\right.$$
$$\left.+\frac{t}{(\alpha+\beta)(\beta+1)}\right]*e(t)$$

(*II*) Consider the positive feedback circuit shown in [4] :



(Fig. 2)

The electric charge $q(t)$ must satisfy the following equation :

$$DL(t)\,Dq(t)+\frac{q(t)}{c(t)}=k(t)\left(e(t)+\frac{q(t)}{c(t)}\right).$$

Suppose $k(t)=k$, $L(t)=at^{\alpha}$ and $c(t)=ct^{1-\alpha}$ with a, $b>0$. Then the previous expression reduces to

$$abA_{\alpha,\beta}\,q_1(t)+(1-k)q_1(t)=e_1(t),$$

where $q(t)=t^{\beta}q_1(t)$ and $e_1(t)=bke(t)t^{1-\alpha-\beta}$. As a consequence, if we demand that $q(0)=0$, then in can be easily obtained that

$$q_1(t)=\frac{1}{abV+(1-k)}\cdot e_1(t)$$

and by invoking (4.5) and the rules (*e*) and (*f*) we obtain the following

$$q_1(t)=\frac{1}{k-1}\left[1-C_{\alpha-1,\beta}\left(\frac{1-k}{ab}t\right)\right]*e_1(t) \quad k\in(0,1)$$

$$q_1(t)=\frac{1}{ab}\,\frac{t}{(\alpha+\beta)\,(\beta+1)}*e_1(t) \qquad k=1$$

$$q_1(t)=\frac{1}{k-1}\left[1-E_{\alpha-1,\beta}\left(\frac{1-k}{ab}t\right)\right]*e_1(t) \quad k>1.$$

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