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ON THE G-FUNCTION OF TWO VARIABLES

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ABSTRACT

In the present paper, certain finite derivatives of the G-function of two variables defined by Agrawal [1], have been evaluated and their applications have been made in summing up the infinite series involving the G-functions of two variables by using the operator $\Omega = x^2 \frac{\partial}{dx}$ studied by Chandel [3, 4].

1. INTRODUCTION.

Agrawal [1] has defined the G-function of two variables in the following way :

$$(1.1) \qquad G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix} x \\ \\ y \end{matrix}\middle| \begin{matrix} (\epsilon_p) \\ (\gamma_t);(\gamma'_t) \\ (\delta_s) \\ (\beta_q);(\beta'_q) \end{matrix}\right]$$

$$= -\frac{1}{4\pi^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} \phi(\xi+\eta)\,\psi(\xi,\eta)\,x^{\xi}\,y^{\eta}\,d\xi\,d\eta,$$

where

$$\phi(\xi+\eta) = \frac{\prod_{j=1}^{n}\Gamma(1-\epsilon_j+\xi+\eta)}{\prod_{j=n+1}^{p}\Gamma(\epsilon_j-\xi-\eta)\prod_{j=1}^{s}\Gamma(\epsilon_j+\xi+\eta)},$$

$\psi(\xi, \eta)=$

$$\frac{\prod_{j=1}^{m_1}\Gamma(\beta_j-\xi)\prod_{j=1}^{m_2}\Gamma(\gamma_j+\xi)\prod_{j=1}^{\nu_1}\Gamma\left(\beta'_j-\eta\right)\prod_{j=1}^{\nu_2}\Gamma\left(\gamma'_j+\eta\right)}{\prod_{j=m_1+1}^{q}\Gamma(1-\beta_j+\xi)\prod_{j=\gamma_1+1}^{t}\Gamma(1-\gamma_j-\xi)\prod_{j=\nu_1+1}^{q}\Gamma\left(1-\beta'_j+\eta\right)\prod_{j=\nu_2+1}^{t}\Gamma\left(1-\gamma'_j-\eta\right)}$$

and $0 \leqslant m_1 \leqslant q$, $0 \leqslant m_2 \leqslant q$; $0 \leqslant \nu_1 \leqslant t$, $0 \leqslant \nu_2 \leqslant t$; $0 \leqslant n \leqslant p$.

The sequence of parameters (β_{m_1}); (β'_{m_2}), (γ_{ν_1}), $(\gamma'_{\nu_2})$and (ϵ_n) are such that none of the poles of the integrand coincide. The path of integration are indented if necessary, in such a manner that all the poles of $\Gamma(\beta_j-\xi)$; $j=1, 2,\ldots m_1$ and $\Gamma(\beta'_\kappa-\eta)$, $k=1, 2,\ldots,m_2$ lie to the right and those of $\Gamma(\gamma_j+\xi)$; $j=1, 2,\ldots\ldots\ldots,\nu_1$ and $\Gamma(\beta'_\kappa-\eta)$, $k=1, 2,\ldots\ldots\ldots, \nu_2$ and $\Gamma(1-\epsilon_j+\xi+\eta)$; $j=1, 2,\ldots\ldots, n$ lie to the left of the imaginary axis. The integral involved are convergent if

$$p+q+s+t < 2(m_1+\nu_1+n)$$
$$p+q+s+t < 2(m_2+\nu_2+n)$$

and

$$|\arg x| < \pi\left[m_1+\nu_1+n-\tfrac{1}{2}(p+q+s+t)\right]; \ |\arg y| < \pi\left[m_2+\nu_2+n-\tfrac{1}{2}(p+q+s+t)\right]$$

The function not only includes Meijer's G-function as it particular case but Appell's functions viz. F_1, F_2, F_3, and F_4 and so many other known functions of one and two variables also.

2. Derivatives of the G-function of two variables :—

In this section, we shall evaluate the rth derivatives of above defined G-function.

First result to be proved is

$$(2.1)\quad \frac{\partial^r}{\partial x^r}\left\{x^{-\beta_1}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix}x\\ \\ y\end{matrix}\middle|\begin{matrix}(\epsilon_p)\\ (\gamma_t);\ (\gamma'_t)\\ (\delta_s)\\ (\beta_q);\ (\beta'_q)\end{matrix}\right]\right\}$$

$$= (-1)^r\, x^{-\beta_1-r}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix}x\\ \\ y\end{matrix}\middle|\begin{matrix}(\epsilon_p)\\ (\gamma_t);(\gamma'_t)\\ (\delta_s)\\ (\beta_1+r, \beta_2,\ldots, \beta_q);\ (\beta'_q)\end{matrix}\right]$$

From (1.1), we have

$$\frac{\partial^r}{\partial x^r}\left\{ x^{-\beta_1}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix}x\\ y\end{matrix}\middle|\begin{matrix}(\epsilon_p)\\ (\gamma_t);(\gamma_t')\\ (\delta_s)\\ (\beta_q);(\beta_q')\end{matrix}\right]\right\}$$

$$=\frac{-1}{4\pi^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\phi(\xi+\eta)\,\psi(\xi,\eta)\,\frac{\partial^r}{\partial x^r}\left(x^{-\beta_1+\xi}\right)y^{\eta}\,d\xi\,d\eta$$

$$=-\frac{1}{4\pi^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}$$

$$\frac{\prod_{j=2}^{m_1}\Gamma(\beta_j-\xi)\prod_{j=1}^{\nu_1}\Gamma(\gamma_j+\xi)\prod_{j=1}^{m_2}\Gamma(\beta_j'-\eta)\prod_{j=1}^{\nu_2}\Gamma(\gamma_j'+\eta)}{\prod_{j=m_1+1}^{q}\Gamma(1-\beta_j+\xi)\prod_{j=\nu_1+1}^{t}\Gamma(1-\gamma_j-\xi)\prod_{j=m_2+1}^{q}\Gamma(1-\beta_j'+\eta)\prod_{j=\nu_2+1}^{t}\Gamma(1-\gamma_j-\eta)}\times$$

$$\times\frac{\prod_{j=1}^{n}\Gamma(1-\epsilon_j+\xi+\eta)\,\Gamma(\beta_1-\xi)}{\prod_{j=n+1}^{p}\Gamma(\epsilon_j-\xi-\eta)\prod_{j=1}^{s}\Gamma(\delta_j+\xi+\eta)}\frac{\partial^r}{\partial x^r}\left(x^{-\beta_1+\xi}\right)y^{\eta}\,d\xi\,d\eta\,.$$

On differentiating r times and then making some adjustments we get the required result.

Similarly, we also obtain the following results :

$$(2.2)\quad \frac{\partial^r}{\partial y^r}\left\{ y^{-\beta'_1}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix}x\\ y\end{matrix}\middle|\begin{matrix}(\epsilon_p)\\ (\gamma_t);(\gamma_t')\\ (\delta_s)\\ (\beta_q);(\beta'_q)\end{matrix}\right]\right\}$$

$$=(-1)^r\, y^{-\beta'_1-r}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix}x\\ y\end{matrix}\middle|\begin{matrix}(\epsilon_p)\\ (\gamma_t);(\gamma_t')\\ (\delta_s)\\ (\beta_q);(\beta'_1+r,\ \beta'_2,\ldots,\beta'_q)\end{matrix}\right],$$

$$(2.3)\quad \frac{\partial^r}{\partial x^r}\left\{ x^{-\gamma_1}\, G^{n,\nu_1,\nu_2,m_1,m_2,}_{p,t,s,q}\left[\begin{matrix}\frac{1}{x}\\ y\end{matrix}\middle|\begin{matrix}(\epsilon_p)\\ (\gamma_t);(\gamma_t')\\ (\delta_s)\\ (\beta_q);(\beta_q')\end{matrix}\right]\right\}$$

$$= (-1)^r \; x^{-\gamma_1 - r} \; G^{n,\, \nu_1,\, \nu_2,\, m_1,\, m_2}_{p,\, t,\, s,\, q} \left[\begin{array}{c|l} \frac{1}{x} & (\epsilon_p) \\ & (\gamma_{1+r}, \gamma_2, \ldots, \gamma_t) \; ; \; (\gamma_t') \\ & (\delta_s) \\ y & (\beta_q) \; ; \; (\beta_q') \end{array} \right]$$

$$(2.4) \quad \frac{\partial^r}{\partial y^r} \left\{ y^{-\gamma'_1} \; G^{n,\, \nu_1,\, \nu_2,\, m_1,\, m_2}_{p,\, t,\, s,\, q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t) \; ; \; (\gamma_t') \\ \frac{1}{y} & (\delta_s) \\ & (\beta_q) \; ; \; (\beta_q') \end{array} \right] \right\}$$

$$= (-1)^r \; y^{-\gamma'_1 - r} \; G^{n,\, \nu_1,\, \nu_2,\, m_1,\, m_2}_{p,\, t,\, s,\, q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t) ; (\gamma'_1 + r, \gamma'_2, \ldots, \gamma'_t) \\ \frac{1}{y} & (\delta_s) \\ & (\beta_q) \; ; \; (\beta_q') \end{array} \right],$$

$$(2.5) \quad \frac{\partial^r}{\partial x^r} \left\{ x^{\beta_q} \; G^{n,\, \nu_1,\, \nu_2,\, m_1,\, m_2}_{p,\, t,\, s,\, q} \left[\begin{array}{c|l} \frac{1}{x} & (\epsilon_p) \\ & (\gamma_t) \; ; \; (\gamma'_t) \\ & (\delta_s) \\ y & (\beta_q) \; ; \; (\beta'_q) \end{array} \right] \right\}$$

$$= (-1)^r \; x^{\beta_q - r} \; G^{n,\, \nu_1,\, \nu_2,\, m_1,\, m_2}_{p,\, t,\, s,\, q} \left[\begin{array}{c|l} \frac{1}{x} & (\epsilon_p) \\ & (\gamma_t) \; ; \; (\gamma'_t) \\ & (\delta_s) \\ y & (\beta_1, \ldots, \beta_{q-1}, \beta_q - r) \; ; \; (\beta'_q) \end{array} \right],$$

$$(2.6) \quad \frac{\partial^r}{\partial y^r} \left\{ y^{\beta'_q} \; G^{n,\, \nu_1,\, \nu_2,\, m_1,\, m_2}_{p,\, t,\, s,\, q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t) \; ; \; (\gamma_t') \\ \frac{1}{y} & (\delta_s) \\ & (\beta_q) \; ; \; (\beta_q') \end{array} \right] \right\}$$

$$= (-1)^r \; y^{\beta_q' - r} \; G^{n,\, \nu_1,\, \nu_2,\, m_1,\, m_2}_{p,\, t,\, s,\, q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t) \; ; \; (\gamma'_t) \\ \frac{1}{y} & (\delta_s) \\ & (\beta_q) \; ; \; (\beta'_1, \ldots, \beta'_{q-1}, (\beta'_q - r) \end{array} \right],$$

$$(2.7) \quad \frac{\partial^r}{\partial x^r} \left[x^{-\gamma_t} \; G^{n,\, \nu_1,\, \nu_2,\, m_1,\, m_2}_{p,\, t,\, s,\, q} \left[\begin{array}{c|l} \frac{1}{x} & (\epsilon_p) \\ & (\gamma_t) \; ; \; (\gamma'_t) \\ & (\delta_s) \\ y & (\beta_q) \; ; \; (\beta'_q) \end{array} \right] \right\}$$

$$= \quad x^{-\gamma_t - r} \; G^{n,\, \nu_1,\, \nu_2,\, m_1,\, m_2}_{p,\, t,\, s,\, q} \left[\begin{array}{c|l} \frac{1}{x} & (\epsilon_p) \\ & (\gamma_1, \ldots, \gamma_{t-1}, \gamma_t + r) \; ; \; (\gamma_t') \\ & (\delta_s) \\ y & (\beta_q) \; ; \; (\beta_q') \end{array} \right],$$

$$(2.8) \quad \frac{\partial^r}{\partial y^r} \left\{ y^{-\gamma'_t} \; G^{n,\, \nu_1,\, \nu_2,\, m_1,\, m_2}_{p,\, t,\, s,\, q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t) \; ; \; (\gamma_t') \\ \frac{1}{y} & (\delta_s) \\ & (\beta_q) \; ; \; (\beta_q') \end{array} \right] \right\}$$

$$= \quad y^{-\gamma'_t - r} \; G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,t,\,s,\,q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t)\,;\,(\gamma'_1, \ldots, \gamma'_{t-1}, \gamma'_t + r) \\ \frac{1}{y} & (\delta_s) \\ & (\beta_q)\,;\,(\beta'_q) \end{array} \right].$$

Particular Cases :

Taking $r=1$, $\gamma_1=0$ and then replacing x by $1/x$ in (2.1), we obtain the known result due to Agrawal [1]

$$x \frac{\partial}{\partial x} \; G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,t,\,s,\,q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_2, \ldots\ldots, \gamma_t);\;(\gamma'_t) \\ & (\delta_s) \\ y & (\beta_q)\,;\,(\beta'_q) \end{array} \right]$$

$$= \; G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,t,\,s,\,q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (1, \gamma_2, \ldots\ldots, \gamma_t);\;(\gamma'_t) \\ & (\delta_s) \\ y & (\beta_q)\,;\,(\beta'_q) \end{array} \right]$$

We have the relation [1; pp. 539]

$$G^{0,\,\nu_1,\,t,\,m_1,\,1}_{0,\,t,\,0,\,q} \left[\begin{array}{c|l} x & \ldots\ldots\ldots\ldots\ldots \\ & (\gamma_t);\;(\gamma'_t) \\ 0 & \ldots\ldots\ldots\ldots\ldots \\ & (\beta_q) \end{array} \right]$$

$$= \frac{\prod_{j=1}^{t} \Gamma(\gamma'_j)}{\prod_{j=2}^{q} \Gamma(1-\beta'_j)} \; G^{m_1,\,\nu_1}_{t,\,q} \left[x \,\middle|\, \begin{array}{l} (1-\gamma_t) \\ (\beta_q) \end{array} \right]; \; (q \geqslant t).$$

Using the above result and giving some particular values to (2.1), (2.3) (2.5) and (2.7), we obtain all the results due to Bhise (2; pp. 351].

3. Summation of Series.

Chandel [3, 4] has used the operators

$$(3.1) \qquad \Omega = x^2 \frac{\partial}{\partial x} \quad \text{and } \omega = y^2 \frac{\partial}{\partial y},$$

to obtain certain operational representations of hypergeometric functions of three variables and certain generating functions for known polynomials.

For the above operator, we can derive

$$(3.2) \qquad e^{t\Omega}\ \{ f(x) \} = f\left(\frac{x}{1-xt}\right),$$

where f (x) has Taylor's series expansion.

Now in this section, we shall make the use of the results obtain in § 2, and the results (3.1) and (3.2) to obtain certain infinite expansions of the G-function of two variables.

Replaçing x by 1/x, in the relation (2.1), we have

$$(-\Omega)^r \left\{ x^{\beta_1}\ G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix}\frac{1}{x}\\ \\ y\end{matrix}\left|\begin{matrix}(\epsilon_p)\\ (\gamma_t);(\gamma'_t)\\ (\delta_s)\\ (\beta_q);(\beta'_q)\end{matrix}\right.\right]\right\}$$

$$= (-1)^r\ x^{\beta_1+r}\ G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix}\frac{1}{x}\\ \\ y\end{matrix}\left|\begin{matrix}(\epsilon_p)\\ (\gamma_t);(\gamma'_t)\\ (\delta_s)\\ (\beta_1+r,\beta_2,\ldots,\beta_q);(\beta_q')\end{matrix}\right.\right]$$

Multiplying both the sides by $\frac{z^r}{r!}$ and summing the expressions from 0 to ∞, we have

$$e^{z\Omega} \left\{ x^{\beta_1}\ G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix}\frac{1}{x}\\ \\ y\end{matrix}\left|\begin{matrix}(\epsilon_p)\\ (\gamma_t);(\gamma'_t)\\ (\delta_s)\\ (\beta_q);(\beta'_q)\end{matrix}\right.\right]\right\}$$

$$= x^{\beta_1}\sum_{r=0}^{\infty}\frac{(xz)^r}{r!}\ G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix}\frac{1}{x}\\ \\ y\end{matrix}\left|\begin{matrix}(\epsilon_p)\\ (\gamma_t);(\gamma'_t)\\ (\delta_s)\\ (\beta_1+r,\beta_2,\ldots,\beta_q);(\beta'_q)\end{matrix}\right.\right].$$

Using the result (3.2), left hand side reduces to

$$\left(\frac{x}{1-xz}\right)^{\beta_1}\ G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix}\frac{1-xz}{x}\\ \\ y\end{matrix}\left|\begin{matrix}(\epsilon_p)\\ (\gamma_t);(\gamma'_t)\\ (\delta_s)\\ (\beta_q);(\beta'_q)\end{matrix}\right.\right]$$

Replacing z by z/x on both the sides, we get on simplification

$$(3.3) \qquad (1-z)^{-\beta_1}\ G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix}x(1-z)\\ \\ y\end{matrix}\left|\begin{matrix}(\epsilon_q)\\ (\gamma_t);(\gamma'_t)\\ (\delta_s)\\ (\beta_q);(\beta'_q)\end{matrix}\right.\right]$$

$$= \sum_{r=0}^{\infty} \frac{z^r}{r!}\, G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,t,\,s,\,q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t)\,;\,(\gamma_t') \\ & (\delta_s) \\ y & (\beta_1+r,\, \beta_1, \ldots, \beta_q)\,;\,(\beta_q') \end{array} \right].$$

Similarly starting from the results (2.2), (2.3), (2,4), (2.5), (2.6), (2.7) & (2.8) : and applying the same technique, we also obtain the following results respectively :

(3.4) $$(1-z)^{-\text{[illegible]}}\; G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,t,\,s,\,q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t)\,;\,(\gamma_t') \\ & (\delta_s) \\ (1-z)\,y & (\beta_q)\,;\,(\beta'_q) \end{array} \right]$$

$$= \sum_{r=0}^{\infty} \frac{z^r}{r!}\, G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,t,\,s,\,q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t)\,;\,(\gamma_t') \\ & (\delta_s) \\ y & (\beta_q)\,;\,(\beta_1'+r,\, \beta_2', \ldots, \beta_q') \end{array} \right],$$

(3.5) $$(1-z)^{-\gamma_1}\; G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,t,\,s,\,q} \left[\begin{array}{c|l} \frac{x}{1-z} & (\epsilon_p) \\ & (\gamma_t)\,;\,(\gamma_t') \\ & (\delta_s) \\ y & (\beta_q)\,;\,(\beta_q') \end{array} \right]$$

$$= \sum_{r=0}^{\infty} \frac{z^r}{r!}\, G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,t,\,s,\,q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_1+r,\, \gamma_2, \ldots, \gamma_t)\,;\,(\gamma_t') \\ & (\delta_s) \\ y & (\beta_q)\,;\,(\beta_q') \end{array} \right],$$

(3.6) $$(1-z)^{-\gamma'_1}\; G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p.\,t,\,s,\,q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t)\,;\,(\gamma_t') \\ \frac{y}{1-z} & (\delta_s) \\ & (\beta_q)\,;\,(\beta_q') \end{array} \right]$$

$$= \sum_{r=0}^{\infty} \frac{z^r}{r!}\, G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,t,\,s,\,q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t)\,;\,\gamma_1'+r,\, \gamma_2', \ldots, \gamma_t' \\ & (\delta_s) \\ y & (\beta_q)\,;\,(\beta_q') \end{array} \right],$$

(3.7) $$(1-z)^{\beta_q}\; G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,t,\,s,\,q} \left[\begin{array}{c|l} \frac{x}{1-z} & (\epsilon_p) \\ & (\gamma_t);\,(\gamma_t') \\ & (\delta_s) \\ y & (\beta_q);\,(\beta_q') \end{array} \right]$$

$$= \sum_{r=0}^{\infty} G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,t,\,s,\,q} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t);\,(\gamma_t') \\ & (\delta_s) \\ y & (\beta_1,\, \beta_2, \ldots\ldots, \beta_{q-1},\, \beta_q-r);\,(\beta_q') \end{array} \right],$$

$$(3.8)\quad (1-z)^{\beta_q'}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix} x \\ \\ \frac{y}{1-z} \end{matrix}\middle|\begin{matrix}(\epsilon_p)\\ (\gamma_t),\ (\gamma_t')\\ (\delta_s)\\ (\beta_q);\ (\beta_q')\end{matrix}\right]$$

$$= \sum_{r=0}^{\infty} \frac{z^r}{r!}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix} x \\ \\ y \end{matrix}\middle|\begin{matrix}(\epsilon_p)\\ (\gamma_t),\ (\gamma_t')\\ (\delta_s)\\ (\beta_q);\ (\beta_1',\ldots,\beta_{q-1}',\ \beta_q'-r)\end{matrix}\right],$$

$$(3.9)\quad (1-z)^{-\gamma_t}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix} \frac{x}{(1-z)} \\ \\ y \end{matrix}\middle|\begin{matrix}(\epsilon_p)\\ (\gamma_t);\ (\gamma_t')\\ (\delta_s)\\ (\beta_q);\ (\beta_q')\end{matrix}\right]$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{z^r}{r!}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix} x \\ \\ y \end{matrix}\middle|\begin{matrix}(\epsilon_p)\\ (\gamma_1,\ldots,\gamma_{t-1},\ \gamma_t+r)\ ;\ (\gamma_t')\\ (\delta_s)\\ (\beta_q)\ ;\ (\beta_q')\end{matrix}\right],$$

$$(3.10)\quad (1-z)^{-\gamma_t'}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix} x \\ \\ \frac{y}{1-z} \end{matrix}\middle|\begin{matrix}(\epsilon_p)\\ (\gamma_t)\ ;\ (\gamma_t')\\ (\delta_s)\\ (\beta_q)\ ;\ (\beta_q')\end{matrix}\right]$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{z^r}{r!}\, G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,q}\left[\begin{matrix} x \\ \\ y \end{matrix}\middle|\begin{matrix}(\epsilon_p)\\ (\gamma_t)\ ;\ (\gamma_1',\ldots,\gamma_{t-1}',\ \gamma_t'+r)\\ (\delta_s)\\ (\beta_q)\ ;\ (\beta_q')\end{matrix}\right].$$

Particular Cases :

We have the result [1, p. 539]

$$G^{1,1,1,1,1}_{1,1,1,1,}\left[\begin{matrix} x \\ \\ y \end{matrix}\middle|\begin{matrix}(\epsilon_1)\\ (\gamma_1)\ ;\ (\gamma_1')\\ (\delta_1)\\ (\beta_1)\ ;\ (\beta_1')\end{matrix}\right] = \frac{x^{\beta_1}\, y^{\beta_1'}\ \Gamma(\gamma_1+\beta_1)\Gamma(\gamma_1'+\beta_1')}{\Gamma(\delta_1+\beta_1+\beta_1')}\times$$

$$F_1\ [\, 1-\epsilon_1+\beta_1+\beta_1'\ ;\ \gamma_1+\beta_1,\ \gamma_1'+\beta_1'\ ;\ \delta_1+\beta_1+\beta_1'\ ;\ -x,\ -y].$$

Now using this result in the result (3.3), we have

$$x^{-\beta_1}\quad y^{\beta_1'}\ \frac{\Gamma(\gamma_1+\beta_1)\ \Gamma(\gamma_1'+\beta_1')}{\Gamma\ (\delta_1+\beta_1+\beta_1')}\ F_1\left[1-\epsilon_1+\beta_1+\beta_1';\ \gamma_1+\beta_1,\ \gamma_1'+\beta_1';\right.$$

$$\left.\delta_1+\beta_1+\beta_1';\ \frac{1-z}{x},\ -y\right]$$

$$\sum_{r=0}^{\infty} \frac{z^r}{r!} \quad x^{-\beta_1 - r} \quad y^{\beta'_1} \frac{\Gamma(\gamma_1+\beta_1+r)\ \Gamma(\gamma'_1+\beta'_1)}{\Gamma(\delta_1+\beta_1+\beta'_1+r)} \times$$

$$F_1 \left[1-\epsilon_1+\beta_1+\beta'_1+r\,;\,\gamma_1+\beta_1+r,\ \gamma_1'+\beta'_1\,;\,\delta_1+\beta_1+\beta'_1+r; -\frac{1}{x}, -y\right].$$

Hence

(3.11) $F_1\ [\,1-\epsilon_1+\beta_1+\beta'_1\,;\,\gamma_1+\beta_1,\ \gamma'_1+\beta'_1;\ \delta_1+\beta_1+\beta'_1\,;\,(z-1)\,x, -y\,]$

$$= \sum_{r=0}^{\infty} \frac{(zx)^r\ \ (\gamma_1+\beta_1)_r}{r!\ (\delta_1+\beta_1+\beta'_1)_r} \times$$

$$F_1\ [\,1-\epsilon_1+\beta_1+\beta'_1+r\,;\,\gamma_1+\beta_1+r,\ \gamma'_1+\beta'_1;\ \delta_1+\beta_1+\beta'_1+r\,;\,-x,\ -y\,].$$

Similarly using the above results, other results for F_1, F_2, F_3 and F_4 can be obtained.

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