Jñānabha, Vol. 1, July 1971.

SOME RULES OF GENERALISED LAPLACE TRANSFORM

By

V. P. SINHA

Department of Mathematics, M. A. College of Technology
Bhopal (M. P.) INDIA

(Received on 20th June, 1971)

1. INTRODUCTION.

The wellknown Laplace transform is defined as

$$(1.1) \qquad F(p) = p \int_0^\infty e^{-pt} f(t)\, dt; \quad R(p) > 0$$

provided that f (t) is sectionally continuous in every finite interval $0 \leq t \leq N$, and for $t > N$, $f(t) = O\, e^{\delta(t)}$, when the integral exists for $R(p) > R(\delta)$, and is symbolically denoted as:

$$F(p) \doteq f(t)$$

F (p), being the image of f (t) in the transform and f (t) is called the inverse transform of F(p) or the original of the image. New operational rules of this transform have been established by various authors by (a) replacing p in F(p), by some suitable function of p and then investigating the corresponding transformation of the original, and (b) replacing t in f(t) by a suitable function of t, and investigating the corresponding transformation of the image Niessen [4] Saxena [5] and Sinha & Saxena [6] have proved some rules of the first kind in Laplace transform.

In this paper, we shall derive rules of the type mentioned in (a) of the above para, proving in each of the generalised Laplace transforms, due to Gupta [3] and Varma [7], which are respectively given by,

$$(1.2) \qquad F(p) = p \int_0^\infty e^{-pt/2} (pt)^{m-1/2} M_{k,m}(pt) f(t) dt, \quad R(p) > 0$$

and

$$(1.3) \qquad F(p) = p \int_0^{\infty} (2pt)^{-\frac{1}{4}} W_{k,m} (2pt) f(t) dt, \quad R(p) > 0.$$

These transforms, shall be symbolically denoted as,

$$F(p) \overset{G}{\underset{k,m}{=\!=}} f(t), \text{ and } F(p) \overset{V}{\underset{k,m}{=\!=}} f(t), \text{ respectively.}$$

First we shall establish, the originals of some binomial functions in these transforms, and the results so obtained will be used in proving the theorems from which a few important deductions are derived, as particular cases of the theorems. The theorems can be widely used in finding the originals of various functions in these transforms. The applications of the theorems shall be illustrated by a few examples.

2. Theorem—1.

If $F(p) \overset{G}{\underset{k,m}{=\!=}} (t)$, then

$$(1.4) \qquad \frac{F[\log (p^n + a^n)]}{\log (p^n + a^n)} \overset{G}{\underset{k,m}{=\!=}}$$

$$\int_0^{\infty} \frac{t^{ns} \Gamma(m+k+\frac{1}{2}) \Gamma(-ns) f(s) \; {}_{n+1}F_{2n} [s, \alpha; \beta; -a^n t^n/n^n]}{\Gamma(2m+1) \Gamma(2m+1+ns) \Gamma(k-m-\frac{1}{2}-ns)} ds$$

where the set of parameters α, β, and γ are given as

$$\alpha \equiv s - \frac{k-m-\frac{3}{2}}{n}, s - \frac{k-m-\frac{5}{2}}{n}, \ldots\ldots\ldots, s - \frac{k-m-\frac{1}{2}-n}{n},$$

$$\beta \equiv s + \frac{1}{n}, s + \frac{2}{n} \ldots\ldots\ldots\ldots\ldots, s + \frac{n}{n}$$

and $$\gamma \equiv s + \frac{2m+1}{n}, s + \frac{2m+2}{n}, \ldots\ldots\ldots\ldots\ldots, s + \frac{2m+n}{n},$$

provided that $R(p) > 0$, $-\frac{1}{2} - R(m) < R(s) < R(k)$, and n is non-negative integer.

Proof :

Substituting $\log (p^n + a^n)$ for p in (1) we have

$$(1.5) \qquad \frac{F[\log (p^n + a^n)]}{\log (p^n + a^n)} = \int_0^{\infty} (p^n + a^n)^{-s} f(s) \, ds$$

Now using [2 ; p. 336], we have

$$(1.6) \quad \int_0^\infty e^{x/2}\, M_{k,m}(x) \,.\, x^{s-1}\, dx = \frac{\Gamma(2m+1)\, \beta(m+\tfrac{1}{2}+s, k-s)}{\Gamma(m+\tfrac{1}{2}-s)}$$

$$-\tfrac{1}{2}-R(m) < R(s) < R(k).$$

Hence, it is easy to see from (1.6), that

$$(1.7) \quad \frac{\Gamma(2m+1)\, \Gamma(2m+\nu+1)\, \Gamma(k-m-\nu-\tfrac{1}{2})}{p^{\nu}\, \Gamma(-\nu)\, \Gamma(m+k+\tfrac{1}{2})} \quad \underset{k,m}{\overset{G}{=}} \quad t^{\nu}$$

provided $-1-2R(m)<R(\nu)<R(k)-R(m)-\tfrac{1}{2}$.

On expanding $(p^n+a^n)^{-s}$ by Binomial theorem, and transforming the expansion term by term with the help of (1.7) and simplifying, we get

$$(1.8) \quad (p^n+a^n)^{-s} \quad \underset{k,m}{\overset{G}{=}} \quad \frac{t^{ns}\, \Gamma(m+k+\tfrac{1}{2})\, \Gamma(-ns)\; {}_{n+1}F_{2n}\, [s, \alpha; \beta, \gamma; -a^n t^n/n^n]}{\Gamma(2m+1)\, \Gamma(2m+1+ns)\, \Gamma(k-m-\tfrac{1}{2}-ns)}$$

On substituting (1.8) in (1.5), we have

$$(1.9) \quad \frac{F[\log(p^n+a^n)]}{\log(p^n+a^n)} = \int_0^\infty f(s)\, ds \,.\, p \int_0^\infty e^{-pt/2}\, (pt)^{m-\frac{1}{2}}\, M_{k,m}(pt) \times$$

$$\frac{t^{ns}\, \Gamma(m+k+\tfrac{1}{2})\, \Gamma(-ns)\; {}_{n+1}F_{2n}\, [s, \alpha; \beta, \gamma; -a^n t^n/n^n]}{\Gamma(2m+1)\, \Gamma(2m+1+ns)\, \Gamma(k-m-\tfrac{1}{2}-ns)}\, dt$$

$$(1.10) = p \int_0^\infty e^{-pt/2}\, (pt)^{m-\frac{1}{2}}\, M_{k,m}(pt)\, dt \int_0^\infty \frac{t^{ns}\, \Gamma(-ns)\, \Gamma(m+k+\tfrac{1}{2})}{\Gamma(2m+1)\, \Gamma(2m+ns+1)}$$

$$\times \frac{1}{\Gamma(k-m-\tfrac{1}{2}-ns)}\; {}_{n+1}F_{2n}\left[s, \alpha; \beta, \gamma; -\frac{a^n t^n}{n^n} \right] f(s)\, ds.$$

on changing the order of integration. On interpreting (1.10) with the help of (1.2), we obtain the required result.

To justify the change of the order of integration we must have besides the conditions stated in (1.1) and (1.4), that f(s) is continuous for $s>0$. As the hypergeometric function ${}_{n+1}F_{2n}$ is convergent, and (1.7) holds for all no-negative values of n, the inner integral is easily seen to be uniformly convergent. Under the stated conditions the resulting s-integral in (1.10) is also convergent. Hence the change of the order of integration is justified by de la valle Poussins theorem [1; p 504].

3. Deductions

(i) On putting n = 1 in (1.4), we have

$$(1.11)\quad \frac{F[\log(p+a)]}{\log(p+a)} \underset{k,m}{\overset{G}{=}} \int_0^\infty \frac{t^s\,\Gamma(m+k+\frac{1}{2})\,\Gamma(-s)\,f(s)}{\Gamma(2m+1)\,\Gamma(2m+1+s)\,\Gamma(k-m-\frac{1}{2}-s)} \times$$

$$ {}_2F_2\,[s, s-k+m+\tfrac{3}{2}\,;\, s+1,\, s+2m+1\,;\, -at]\,ds.$$

Substituting m = 0, k = ½, (1.11) reduces to

$$(1.12)\qquad \frac{F[\log(p+a)]}{\log(p+a)} \doteq \int_0^\infty \frac{t^s\,f(s)}{\Gamma(1+s)}\,{}_1F_1\,[s\,;\,s+1\,;-at]\,ds.$$

By putting a = 0, (1.12) reduces to

$$(1.13)\qquad \frac{F[\log p]}{\log p} \doteq \int_0^\infty \frac{t^s}{\Gamma(1+s)}\,.\,f(s)\,ds.$$

(ii) Again putting m = 0, k = ½ in (1.4), we have

$$(1.14)\quad \frac{F[\log(p^n+a^n)]}{\log(p^n+a^n)} \doteq \int_0^\infty \frac{t^{ns}\,f(s)}{\Gamma(1+ns)}\,{}_1F_n\left[s\,;\,s+\frac{1}{n},\ldots,s+\frac{n}{n};-\frac{a^n t^n}{n^n}\right]ds.$$

The particular case of Theorem-1 given by (1.14) have been proved by Saxena [5] as a theorem. Result (1.13) is well known [2, p. 132] and results (1.11) and (1.12) are believed to be new.

4. Example 1.

If f(t) is defined as

$$f(t) = \begin{cases} 0, & 0<t<c \\ 1, & c<t<b \\ 0, & t>b \end{cases}\quad ,\ \text{for } 0\leq c<b$$

We have [2, p. 241]

$$\left(e^{-cp} - e^{-bp}\right) \doteq f(t)$$

Using (1.4), we obtain,

$$(1.15)\qquad \frac{[(p^n+a^n)^{-c}-(p^n+a^n)^{-b}]}{\log(p^n+a^n)}$$

$$\underset{k,m}{\overset{G}{=}} \int_c^b \frac{t^{ns}\,\Gamma(m+k+\frac{1}{2})\Gamma(-ns)\,{}_{n+1}F_{2n}\left[s,\alpha;\beta,\gamma;-\frac{a^n t^n}{n^n}\right]}{\Gamma(2m+1)\,\Gamma(2m+1+ns)\,\Gamma(k-m-\frac{1}{2}-ns)}\,ds.$$

Substituting m = 0, k = ½ in (1.15), we get

$$(1.16)\quad \frac{[(p^n+a^n)^{-a}-(p^n+a^n)^{-b}]}{\log(p^n+a^n)} \doteq \int_c^b \frac{t^{ns}}{\Gamma(ns+1)}$$

$$ {}_1F_n\left[s;\ s+\frac{1}{n},\ \ldots\ldots,\ s+\frac{n}{n};\ -\frac{a^n t^n}{n^n}\right]ds$$

Example 2.

If f(t) is defined as

$$f(t)=\begin{cases}0, & 0<t<b\\ 1, & t>b\end{cases}\quad,\quad \text{for } b>0.$$

We have [2, p. 241]

$$e^{-bp} \doteq f(t).$$

Using (1.4), we get

$$(1.17)\quad \frac{(p^n+a^n)^{-b}}{\log(p^n+a^n)}$$

$$\frac{G}{k,m}\int_b^{\infty}\frac{t^{ns}\,\Gamma(m+k+\frac{1}{2})\,\Gamma(-ns)\ {}_{n+1}F_{2n}\left[s,\alpha;\beta,\gamma;-\frac{a^n t^n}{n^n}\right]}{\Gamma(2m+1)\,\Gamma(2m+1+ns)\,\Gamma(k-m-\frac{1}{2}-ns)}ds,$$

which on putting n = 1, a = 0, reduces to

$$(1.18)\quad \frac{p^{-b}}{\log p}\ \frac{G}{k,m}\int_b^{\infty}\frac{t^{s}\,\Gamma(m+k+\frac{1}{2})\,\Gamma(-s)}{\Gamma(2m+1)\,\Gamma(2m+1+s)\,\Gamma(k-m-\frac{1}{2}-s)}ds.$$

Putting m = 0, k = ½ in (1.18), we have

$$(1.19)\quad \frac{p^{-b}}{\log p}\doteq\int_b^{\infty}\frac{t^{s}}{\Gamma(1+s)}ds.$$

5. **THEOREM 2.**

If $F(p)\ \frac{V}{k,m}\ f(t)$ then

$$(1.20)\quad \frac{F[\log(p^n+a^n)]}{\log(p^n+a^n)}\ \frac{V}{k,m}$$

$$\int_0^{\infty}\frac{t^{ns}\,\Gamma\left(ns-k-\frac{7}{4}\right)I_{m+ns}}{2^{m+\frac{1}{4}}\,\Gamma\left(m+ns+\frac{5}{4}\right)\Gamma\left(-m+ns+\frac{5}{4}\right)\ {}_2F_1\left\{\begin{matrix}m,\ ns+\frac{5}{4},\ m-k+\frac{1}{2};-1\\ ns-k+\frac{7}{4};\end{matrix}\right\}}ds$$

where $I_{m,ns}$ stands for the infinite series

$$\equiv \left(1 - \frac{s\, a^n\, t^n \Gamma(ns-k-\frac{7}{4}+n)\,\Gamma(m+ns+\frac{5}{4})\,\Gamma(-m+ns+\frac{5}{4})}{\Gamma(ns-k+\frac{7}{4})\Gamma(m+ns+\frac{5}{4}+n)\Gamma(-m+ns+\frac{5}{4}+n)} \times \frac{{}_2F_1[m+ns+\frac{5}{4};\, m-k+\frac{1}{2};\, ns-k+\frac{7}{4};\, -1]}{{}_2F_1[m+ns+n+\frac{5}{4},\, m-k+\frac{1}{2};\, ns+n-k+\frac{7}{4};-1]} + \ldots\ldots\infty \right)$$

provided $R(p) > 0$, $R(s) > |R(m)| - \frac{1}{2}$, and n is a non-negative integer.

Proof :—Substituting $\log_e(p^n+a^n)$ for p in (1.1), we get (1.5).
Now using [2, p. 337], we have

$$(1.21) \quad \int_0^\infty x^{s-1} W_{k,m}(x)\,dx = \frac{2^{m+s+\frac{1}{2}}\,\Gamma(m+s+\frac{1}{2})\,\Gamma(-m+s+\frac{1}{2})}{\Gamma(s-k+\frac{7}{4})}\ {}_2F_1\left[\begin{matrix} m+s+\frac{1}{2},\, m-k+\frac{1}{2};-1 \\ s-k+1 \end{matrix}\right],$$

$$R(s) > |R(m)| - \tfrac{1}{2}.$$

It is easy to see from (1.21) that

$$(1.22) \quad \frac{2^{m+\frac{1}{4}}\,\Gamma(m+\nu+\frac{5}{4})\,\Gamma(-m+\nu+\frac{5}{4})\ {}_2F_1\left(\begin{matrix} m+\nu+\frac{5}{4},\, m-k+\frac{1}{2}\,;\,-1 \\ \nu-k+\frac{7}{4}\,; \end{matrix}\right)}{p^\nu\ \Gamma(\nu-k+\frac{7}{4})} \ \frac{V}{\overline{k,\,m}}\ t^\nu,$$

provided $R(\nu) > -\frac{5}{4}$.

Expanding $(p^n+a^n)^{-s}$ with the help of Binomial theorem, and transforming the expansion term by term by using (1.22), and simplifying we get

$$(1.23) \quad (p^n+a^n)^{-s}\ \frac{V}{\overline{k,m}}\ \frac{t^{ns}\,\Gamma(ns-k+\frac{7}{4})\,I_{m,ns}}{2^{m+\frac{1}{4}}\,\Gamma(m+ns+\frac{5}{4})\,\Gamma(-m+ns+\frac{5}{4})\ {}_2F_1\left(\begin{matrix} m+ns+\frac{5}{4},\, m-k+\frac{1}{2}\,;\,-1 \\ ns-k+\frac{7}{4} \end{matrix}\right)}$$

proceeding on the lines of the proof of Theorem 1, we can arrive at (1.20), the required result.

6. DEDUCTIONS.

(i) Putting $a = 0$ in (1.20), we have

$$(1.14)\qquad \frac{F\,[n \log p]}{n \log p}\ \overline{\overline{\frac{V}{k,m}}}\ \int_0^{\infty} \frac{t^{ns}\,\Gamma(ns-k+\frac{7}{4})}{2^{m+\frac{1}{4}}\ \Gamma(m+ns+\frac{5}{4})\ \Gamma(-m+ns+\frac{5}{4})}\ {}_2F_1\left[\begin{matrix} m+ns+\frac{5}{4},\ m-k+\frac{1}{2};-1 \\ ns-k+\frac{7}{4} \end{matrix}\right] ds$$

(ii) Substituting $m = -\frac{1}{4}$, $k = \frac{1}{4}$ in (1.20), reduces on simplification to (1.14).

Again substituting $m = -\frac{1}{4}$, $k = \frac{1}{4}$, and $a = 0$, (1·20) reduces to

$$(1.25)\qquad \frac{F\,[n \log p]}{n \log p}\ \doteqdot\ \int_0^{\infty} \frac{t^{ns}\,f(s)}{\Gamma(ns+1)}\,ds.$$

7. **Example :**

If $f(t)$ is defined as

$$f(t) = \begin{cases} 1, & 0<t<b \\ 0, & t>b \end{cases}, \text{ for } b>0$$

we have [2, p. 241], $(1-e^{-bp}) \doteqdot f(t)$.

Using (1.20), we have

$$(1.26)\qquad \frac{[1-(p^n+a^n)^{-b}]}{\log(p^n+a^n)}\ \overline{\overline{\frac{V}{k,m}}}\ \int_0^{b} \frac{t^{ns}\,\Gamma(ns-k+\frac{7}{4})\ I_{m,ns}}{2^{m+\frac{1}{4}}\ \Gamma(m+ns+\frac{5}{4})\ \Gamma(-m+ns+\frac{5}{4})}\ {}_2F_1\left[\begin{matrix} m+ns+\frac{5}{4}, m-k+\frac{1}{2};-1 \\ ns-k+\frac{7}{4}; \end{matrix}\right] ds.$$

Putting $a = 0$, $n = 1$ in (1.26), we get

$$(1.27)\qquad \frac{1-p^{-b}}{\log p}\ \overline{\overline{\frac{V}{k,m}}}\ \int_0^{b} \frac{t^{s}\,\Gamma(s-k+\frac{7}{4})}{2^{m+\frac{1}{4}}\ \Gamma(m+s+\frac{5}{4})\ \Gamma(-m+s+\frac{5}{4})}\ {}_2F_1\left[\begin{matrix} m+s+\frac{5}{4},\ m-k+\frac{1}{2};-1 \\ s-k+\frac{7}{4} \end{matrix}\right] ds$$

which on putting $m = -\frac{1}{4}$, $k = \frac{1}{4}$ reduces to,

$$(1.28)\qquad \frac{1-p^{-b}}{\log p}\ \doteqdot\ \int_0^{b} \frac{t^{s}}{\Gamma(s+1)}\,ds.$$

REFERENCES

[1] Bromwich, T. J. I. A; 1959 : 'An introduction to the theory of Infinite series' Macmillan.

[2] Erdelyi, A. ; 1954 : 'Tables of Integral Transforms' Vol. 1 Mc-Graw Hill.

[3] Gupta, P. M. ; 1962 : 'Some problems in operational Calculus Agra University Journal of Research Vol. XI Part II pp. 59-79.

[4] Neissen, K. F. ; 1935 : Phil. Mag. XX p. 977.

[5] Saxena, V. P. ; 1967 : 'On some rules of operational Calculus' Ganita, Vol. 18 p. 17.

[6] Sinha, V. P. and Saxena V. P. 1970 : 'On some rules of Meijer and Varma transform', (under publication).

[7] Varma, R. S. ; 1947 : 'Generalisation of Laplace Transform', Current Science Vil. 16 p. 17.