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ON THE STRONG LOGARITHMIC SUMMABILITY OF A FACTORED FOURIER SERIES

By

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1. Let f (θ) be integrable (L) over $(-\pi, \pi)$ and be periodic outside this interval with period 2π.

Let the Fourier series of f (θ) be

$$(1.1) \quad \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \operatorname{Cos} n\theta + b_n \operatorname{Sin} n\theta) = \tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n(\theta).$$

Let $\phi(u) = \tfrac{1}{2}[f(\theta+u) + f(\theta-u) - 2S]$,

$$S = S(\theta) \quad \text{and } S_n = S_n(\theta) = \tfrac{1}{2} A_0 + \sum_{\nu=1}^{n} A_\nu(\theta).$$

2. Hardy and Littlewood [1] have shown that the Fourier series of f (θ) is not strongly summable in the Lebesgue set. They have proved that

Theorem A. If $\int_0^t |\phi(u)| \, du = O(t)$,

then $$\sum_{\nu=0}^{n} (S_\nu - S)^2 = O(n \log n),$$

and $$\sum_{\nu=0}^{n} |S_\nu - S| = O(n \sqrt{\log n}).$$

In the same paper they have also mentioned the fact that the Fourier series of f (θ) is not strongly summable by logarithmic means in the Lebesgue set. B. Deokinandan [2] considered the strong logarithmic summability of the Fourier series in the Lebesgue set and proved the following theorem.

Theorem B. If $\int_0^t |\phi(u)|\, du = O(t)$,

$$\text{then} \quad \sum_{m=1}^{n} \frac{|S_m - S|}{m} = O(\log^{3/2} n) .$$

Pati [3] has raised a problem whether the

series $\sum \frac{|S_\nu - f(\theta)|\, \lambda_\nu}{\nu}$ is convergent for every (or atleast, some) convex sequence $\{\lambda_n\}$ such that $\Sigma\, n^{-1} \lambda_n < \infty$

where $\int_0^t |\phi^*(u)|\, du = o(t)$,

and $\phi^*(u) = \frac{1}{2}[f(\theta+u) + f(\theta-u) - f(\theta)]$.

In the present note we consider the strong logarithmic summability of a factored Fourier series and prove a theorem which gives an estimate of the n th partial sum of the series

$$\sum_{\nu=1}^{n} \frac{|S_\nu - S|\, \lambda_\nu}{\nu}$$

under the conditions mentioned in Pati's paper.

In fact we prove the following theorem.

Theorem. If $\int_0^t |\phi(u)|\, du = 0(t)$,

and $\{\lambda_n\}$ is a convex and bounded sequence such that $\Sigma n^{-1}\lambda_n$ is convergent then

$$\sum_{\nu=1}^{n} \frac{|S_\nu - S|\, \lambda_\nu}{\nu} = 0(\log n)$$

3. In order to prove the theorem, we require the following Lemma :

Lemma* : If $\{\lambda_n\}$ is a convex and bounded sequence such that $\Sigma n^{-1}\lambda_n$ is convergent, then λ_n is non-negative and non-increasing, $n\Delta\lambda_n = 0(1)$ and $\lambda_n \log n = 0(1)$, as $n \to \infty$, where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

4. **Proof of the Theorem** :—We consider only the reduced case in which $f(\theta) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta$ is an even function and suppose that particular values of θ to be considered is zero and the sum of the series for $\theta = 0$ is (if it exists) also zero. So that

$$\phi(u) = f(u),$$

$$S_n = \frac{1}{2} a_0 + \sum_{\nu=1}^{n} a_\nu .$$

If the theorem is true in this case it will (as usual in the Theory of Fourier series) be true generally.

We write

$$f(z) = \sum_{\nu=0}^{\infty} C_\nu z^\nu,$$

$$C_0 = \frac{1}{2} a_0, \; C_\nu = a_\nu \qquad (\nu > 0).$$

Denote the n th partial sum of $\sum_{\nu=0}^{\infty} C_\nu z^\nu$ at $z=1$ by S_n, i.e., $S_n = \sum_{\nu=0}^{n} C_\nu$

and the n th partial sum of

$$f'(z) = \sum_{\nu=0}^{\infty} \nu C_\nu z^{\nu-1}$$

*Chow H. C. [4] (Lemma 3 and 4).

at $z=1$ by $S_n(f')$ i.e.,

$$S_n(f') = \sum_{\nu=0}^{n} \nu C_\nu = \sum_{\nu=1}^{n} \nu C_\nu ,$$

where the dash denotes differentiation with respect to z.

Denote the (C,1) means of $f(z) = \sum_{\nu=0}^{\infty} C_\nu z^\nu$ at $z=1$

by $\sigma_n = \sigma_n(f)$ i.e.,

$$\sigma_n = \sigma_n(f) = \frac{S_0+S_1+\ldots+S_n}{n+1} = \sum_{\nu=0}^{n} \left\{1-\frac{\nu}{n+1}\right\} C_\nu ,$$

So that

$$S_n - \sigma_n = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu C_\nu = \frac{S_n(f')}{n+1} .$$

Now

(4.1)
$$P = \sum_{\nu=1}^{n} \frac{|S_\nu - S|\lambda_\nu}{\nu}$$

$$= \sum_{\nu=1}^{n} \frac{|S_\nu - \sigma_\nu + \sigma_\nu - S|\lambda_\nu}{\nu}$$

$$\leqslant \sum_{\nu=1}^{n} \frac{|S_\nu - \sigma_\nu|\lambda_\nu}{\nu} + \sum_{\nu=1}^{n} \frac{|\sigma_\nu - S|\lambda_\nu}{\nu}$$

$$= P_1 + P_2 \quad , \text{say.}$$

Since by hypothesis $\sigma_n \to S$, it follows that

(4.2) $$P_2 = \sum_{\nu=1}^{n} \frac{|\sigma_\nu - S|\lambda_\nu}{\nu}$$

$$= O\left(\sum \frac{\lambda_\nu}{\nu}\right) = O(1).$$

Again $$P_1 = \sum_{\nu=1}^{n} \frac{|S_\nu - \sigma_\nu|\,\lambda_\nu}{\nu} = \sum_{\nu=1}^{n} \frac{|S_\nu(f')|\,\lambda_\nu}{\nu(\nu+1)}$$

$$< \sum_{\nu=1}^{n} \frac{|S_\nu(f')|\,\lambda_\nu}{\nu^2}$$

so that by Schwartz's inequality we have

$$(4.3)\qquad P_1 < \left(\sum_{\nu=1}^{n} \frac{|S_\nu(f')|^2 \lambda_\nu}{\nu^3}\right)^{1/2} \cdot \left(\sum_{\nu=1}^{n} \frac{\lambda_\nu}{\nu}\right)^{1/2}$$

$$= O\left(\sum_{\nu=1}^{n} \frac{|S_\nu(f')|^2 \lambda_\nu}{\nu^3}\right)^{\frac{1}{2}}$$

$$= O\left(Q^{\frac{1}{2}}\right), \quad \text{say}$$

where $$Q = \sum_{\nu=1}^{n} \frac{|S_\nu(f')|^2 \lambda_\nu}{\nu^3}$$

From Abel's Transformation we get

$$Q = \sum_{\nu=1}^{n-1} \left\{\sum_{m=1}^{\nu} \left(S_m(f')\right)^2\right\} \left(\frac{\lambda_\nu}{\nu^3} - \frac{\lambda_{\nu+1}}{(\nu+1)^3} + \frac{\lambda_n}{n^3}\right).$$

$$\left\{\sum_{\nu=1}^{n} \left(S_\nu(f')\right)^2\right\}$$

Now

$$\frac{\lambda_\nu}{\nu^3} - \frac{\lambda_{\nu+1}}{(\nu+1)^3} = \frac{\nu^3\left(\lambda_\nu - \lambda_{\nu+1}\right) + \lambda_\nu\,(3\nu^2 + 3\nu + 1)}{\nu^3(\nu+1)^3}$$

$$= \frac{\nu^3 \Delta \lambda_\nu + \lambda_\nu (3\nu^2+3\nu+1)}{\nu^3(\nu+1)^3}$$

$$= \frac{\nu^3 . O\left(\frac{1}{\nu}\right) \lambda_\nu + O(\nu^2)}{\nu^6}$$

$$= O\left(\frac{1}{\nu^4}\right) + O\left(\frac{\lambda_\nu}{\nu^4}\right)$$

Therefore

$$Q = \sum_{\nu=1}^{n-1} \left\{ \sum_{m=1}^{\nu} \left(S_m(f')\right)^2 \right\} . \left\{ O\left(\frac{1}{\nu^4}\right) + O\left(\frac{\lambda_\nu}{\nu^4}\right) \right\}$$

$$+ \frac{\lambda_\nu}{n^3} \left\{ \sum_{\nu=1}^{n-1} \left(S_\nu (f') \right)^2 \right\}$$

$$= \sum_{\nu=1}^{n-1} \left\{ \sum_{m=1}^{\nu} \left(S_m (f')\right)^2 \right\} . O\left(\frac{1}{\nu^4}\right) + O\left[\sum_{\nu=1}^{n-1} \left\{ \sum_{m=1}^{\nu} \left(S_m (f')\right)^2 \right\} \right]$$

$$+ \frac{\lambda_n}{n^3} \left[\sum_{\nu=1}^{n} \left(S_\nu (f') \right)^2 \right].$$

Hardy and Littlewood [1] have proved under the condition

$$\int_0^t | \phi (4) | d_4 = O (t)$$

that

$$\sum_{\nu=o}^{n} \left(S_\nu (f') \right)^2 = O (n^3 \log n).$$

So that we have

$$Q = O\left(\sum_{\nu=1}^{n} \frac{\nu^3 \log \nu}{\nu^4} \right) + O\left(\sum_{\nu=1}^{n} \frac{\nu^3 \lambda\nu \log \nu}{\nu^4} \right) + O\left(\frac{\lambda_n}{n^3} n^3 \log n \right)$$

$$= O\left(\sum_{\nu=1}^{n} \frac{\log \nu}{\nu}\right) + O\left(\sum_{\nu=1}^{n} \frac{\lambda_\nu \log \nu}{\nu}\right) + O(\lambda_n \log n)$$

$$= O(\log^2 n) + O(\log n) + O(1).$$

Hence

(4.4) $\quad Q = O(\log^2 n).$

From (4.3) and (4.4), we obtain

(4.5) $\quad P_1 = O(\log n).$

Finally, in virtue of (4.1), (4.2) and (4.5) we have

$$P = O(\log n).$$

This completes the proof of the theorem.

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