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ON A NEW CLASS OF KERNELS†

By

VINOD PRAKASH SAXENA

Department of Mathematics M. A. College of Technology, BHOPAL-7*

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1. INTRODUCTION

A Kernel K (x), by means of which under suitable conditions an arbitrary function f (x) is capable of being represented as a repeated integral

$$(1.1) \qquad f(x) = \int_0^\infty K(ux) \int_0^\infty K(uy)\, f(y)\, dy\, du \quad ,$$

will be called as a symmetrical Fourier Kernel.

The pair of Kernels K_1 (x) and K_2 (x), by means of which under appropriate conditions, the function f (x) is represented as

$$(1.2) \qquad f(x) = \int_0^\infty K_1(ux) \int_0^\infty K_2(uy)\, f(y)\, dy\, du$$

will be called as a pair of unsymmetrical Fourier Kernels. K_1 (x) and K_2 (x) will also be known as the reciprocal Kernels of each other. The convergence conditions of the relations (1.1) and (1.2) are formulated in two theorems of Hardy and Titchmash [1].

In a recent paper, Saxena & Agrawal [2] have generalized the formula (1.1), assuming the Kernel K (x) in the form of a finite sum of symmetrical

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*Present Address : S.V. Regional College of Engg. & Tech., Surat-1.

Fourier Kernels. Generalising this, and the formula (1.2) as well, we shall establish here the following theorem :

2. THEOREM

Let K (x) be a Kernel function expressed as

(2.1) $K(x) = \sum_{r=1}^{n} K_r(x)$, n is finite, and $\overline{K}_r(s)$, the Mellin transform of $K_r(x)$, where $\overline{L}_r(s) = 1/\overline{K}(1-s)$, which satisfy the conditions :

(i) $\overline{K}_r(s)$ and $\overline{L}_r(s)$, $s = v + it$ (v and t are real) are regular in the strips $\nu_1 < \nu < \nu_2$, $\nu'_1 < \nu < \nu'_2$ where $\nu_1, \nu'_1 < 0$; $\nu_2, \nu'_2 > 1$ except perhaps for a finite number of simple poles on the imaginary axis,

(ii) $$\overline{K}r(s) = \{A_{1,r} + B_{1,r}/s + O(1/|s|^2)\}\,\Gamma(s)\cos\tfrac{1}{2}s\pi,\ t\to\infty$$

$$= \{A_{2,r} + B_{2,r}/s + O(1/|s|^2)\}\,\Gamma(s)\cos\tfrac{1}{2}s\pi,\ t\to-\infty$$

$$\overline{L}r(s) = \{C_{1,r} + D_{1,r}/s + O(1/|s|^2)\}\,\Gamma(s)\cos\tfrac{1}{2}s\pi,\ t\to\infty$$

$$= \{C_{2,r} + D_{2,r}/s + O(1/|s|^2)\}\,\Gamma(s)\cos\tfrac{1}{2}s\pi,\ t\to-\infty$$

where $A_{1,r}$, $B_{1,r}$, $A_{2,r}$, $B_{2,r}$, $C_{1,r}$, $D_{1,r}$, $C_{2,r}$ and $D_{2,r}$ (r=1, 2,..................n) are all constants,

(iii) f(u) and h(u) functions of bounded variation near u = x, $f(x) \in L(0, \infty)$ and $f(s) \in L(\tfrac{1}{2}-i\infty, \tfrac{1}{2}+i\infty)$, then, solution of the integral equation

(2.2) $$f(x) = \int_0^\infty K(ux)\,h(u)\,du$$

is given by

(2.3) $$\tfrac{1}{2}\{h(x+o)+h(x-o)\} = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \phi(s)\,\overline{f}(1-s)\,x^{-s}\,ds,$$

where

(2.4) $$\phi(s) = \left[\sum_{r=1}^{n} 1/\overline{L}_r(s)\right]^{-1},$$

$\overline{f}(s)$ is Mellin transform of f (x) .

Proof :—

Let us assume

$$(2.5)\quad \int_0^\infty K_r\,(ux)\,h\,(u)\,du = f_r\,(x) \qquad (r = 1, 2, \ldots\ldots\ldots, n)$$

so that

$$(2.6)\qquad f\,(x) = \sum_{r=1}^{n} f_r\,(x).$$

It is easy to understand from the conditions prescribed in the theorem and from the formula of Hardy and Titchmarsh [1], that $K_r\,(x)$ and $L_r\,(x)$ are reciprocal Kernels.

Hence

$$(2.7)\quad h\,(x) = \int_0^\infty L_r\,(ux)\,f_r\,(u)\,du \qquad (r = 1, 2, \ldots\ldots\ldots, n)$$

Multiplying both the sides of this equation by x^{s-1} and integrating, under the integral sign, with respect to x from 0 to ∞, we get

$$(2.8)\qquad \bar{h}\,(s) = \bar{L}_r\,(s)\,\bar{f}_r\,(1-s) \qquad (r = 1, 2, \ldots\ldots\ldots, n)$$

where $\bar{h}\,(s)$ and $\bar{f}_r\,(s)$ are Mellin transforms of $h\,(x)$ and $f_r\,(x)$ respectively.

Similarly, from (2.6), we deduce

$$(2.9)\quad \bar{f}(s) = \sum_{r=1}^{n} \bar{f}_r\,(s)\ .$$

Therefore, from (2.8) and (2.9), we get

$$(2.10)\quad \bar{h}\,(s) = \phi\,(s)\,\bar{f}\,(1-s)$$

On applying Mellin inversion formula to (2.10), we arrive at the theorem.

Convergence Conditions :

Now we give justification of our analysis by considering convergence conditions of the integrals and the functions involved.

Fox's lemma [3] states that :

If the functions $j(x)$ and $i(x)$ belong to the class $L(o, \infty)$ then, the functions $g(x)$, defined by

$$(2.11) \qquad g(x) = \int_{0}^{\infty} j(ux)\, i(u)\, du$$

belongs to $L(o, \infty)$"

Hence for $Re(s) = \frac{1}{2}$, we find that the functions $h(x)$ and $K_r(x)$ satisfy the conditions of Hardy and Titchmarsh's theorem. Therefore, the inversion of (2.5) is valid. Also the integrals involved during the steps (2.7) and (2.8) are absolutely covergent, which implies that the integration under the sign of integral in (2.7) is permissible.

Also, $\phi(s)$ can be simplified as

$$\phi(s) = \prod_{r=1}^{n} \overline{L}_r(s) \Big/ \left\{ \prod_{r=2}^{n} \overline{L}_r(s) + \overline{L}_1(s) \prod_{r=3}^{n} \overline{L}_r(s) + \ldots + \prod_{r=1}^{n-1} \overline{L}_r(s) \right\}$$

Substituting the value of $\overline{L}_r(s)$ $(r = 1, 2, \ldots\ldots, n)$ from the condition (ii) of the Theorem, collecting terms of equal powers of s and dividing the expression thus obtained, we find

$$(2.12) \qquad \phi(s) = \{ E_1 + F_1/s + O(1/|s|^2) \} \, \Gamma(s) \cos \tfrac{1}{2} s\pi, \ t \to \infty$$

$$= \{ E_2 + F_2/s + O(1/|s|^2) \} \, \Gamma(s) \cos \tfrac{1}{2} s\pi, \ t \to -\infty$$

where E_1, F_1, E_2, F_2 are constants depending on $C_{1,r}$, $D_{1,r}$, $C_{2,r}$ and $D_{2,r}$.

Hence if $\overline{f}(s) \in L(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and it is of bounded variation throughout the interval, then

$$\overline{h}(s) \in L(\tfrac{1}{2} - i\infty, \tfrac{1}{2} + i\infty) .$$

Therefore, the Mellin inversion formula is applicable to $h(s)$ due to known therorem [4, p. 46].

3. Examples :

(i) We take

$$K(x) = \sqrt{\frac{x}{2}} \left[J_\nu(x) + Y_\nu(x) \right] = \sum_{r=1,\,2} K_r(x)$$

So that using the Mellin transforms of Bessel functions, and the formula $\Gamma(\frac{1}{2}-z)\,\Gamma(\frac{1}{2}+z) = \pi \sec \pi z$, we obtain from (2.4)

$$\phi(s) = 2^{1-s}\left[\frac{\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}s)}+\frac{\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)\,\Gamma(\frac{3}{4}-\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(\frac{3}{4}+\frac{1}{2}\nu+\frac{1}{2}s)\,\Gamma(\frac{1}{4}-\frac{1}{2}\nu-\frac{1}{2}s)}\right]$$

$$= \frac{2^{1-s}\,\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)\,\Gamma(\frac{3}{4}-\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)\,\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}s)}.$$

Hence, from the theorem we find that the solution of the intergral equation

$$(3.1)\quad f(x) = \frac{1}{\sqrt{2}}\int_0^\infty (ux)^{\frac{1}{2}}\,[J_\nu(ux)+Y_\nu(ux)]\,h(u)du$$

is given by

$$(3.2)\quad h(x) = \frac{1}{\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\Gamma(\frac{3}{4}+\frac{1}{2}\nu-\frac{1}{2}s)\,\Gamma(\frac{3}{4}-\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)\,\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}s)}\,(2x)^{-s}\,\bar{f}(1-s)\,ds$$

This last relation, using Parseval's theorem of Mellin transform, reduces to

$$(3.3)\quad h(x) = \int_0^\infty G^{1,1}_{1,3}\left(\frac{u^2x^2}{4}\,\middle|\,\begin{matrix}\frac{1}{4}+\frac{1}{2}\nu\\ \frac{1}{4}+\frac{1}{2}\nu,\ \frac{1}{2}\nu,\ -\frac{1}{2}\nu\end{matrix}\right)f(u)du$$

provided h (u) is continuous at u=x. This result is in accordance with known result of Hardy [5].

(ii) Similarly, if

$$(3.4)\qquad K(x) = K_1(x)+K_2(x)$$

where

$$K_1(x) = \frac{1}{4}\,x^{3/2}\,G^{2,0}_{0,4}\left(\left(\frac{x}{4}\right)^4\,\middle|\,-\frac{3}{8},-\frac{1}{8},\frac{3}{8},\frac{1}{8}\right)$$

$$K_2(x) = \frac{1}{4}\,x^{3/2}\,G^{2,0}_{0,4}\left(\left(\frac{x}{4}\right)^4\,\middle|\,\frac{1}{8},\frac{3}{8},-\frac{1}{8},-\frac{3}{8}\right)$$

then using Mellin transform for the G-functions and using well known formulae of Gamma functions, we obtain

$$(3.5)\qquad \phi(s) = \frac{2}{\sqrt{\pi}}\,\Gamma(s)\sin s\pi.$$

Hence we obtain for h (u), continuous at u=x,

$$\text{(3.6)} \quad h(x) = \lim_{T\to\infty} \frac{1}{i\,\pi^{3/2}} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \Gamma(s) \sin s\pi\, \bar{F}(1-s)\, x^{-s}\, ds$$

$$= \frac{1}{4i\sqrt{\pi}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} e^{zx} f(z)\, dz$$

Now if we use the identities (6) :

$$\text{(3.7)} \quad G^{2,0}_{0,4}\left(\left(\frac{x}{4}\right)^4 \middle| -\frac{3}{8}, -\frac{1}{8}, \frac{3}{8}, \frac{1}{8}\right) = \frac{4}{\sqrt{\pi}} x^{-3/2} \left(e^{-x} + \cos x - \sin x\right)$$

$$\text{(3.8)} \quad G^{2,0}_{0,4}\left(\left(\frac{x}{4}\right)^4 \middle| \frac{1}{8}, \frac{3}{8}, -\frac{1}{8}, -\frac{3}{8}\right) = \frac{4}{\sqrt{\pi}} x^{-3/2} \left(e^{-x} - \cos x + \sin x\right)$$

then we again arrive at the result (3.6).

In this way we can construct numerous examples giving inversion formulae to various integral equations.

4. Particular Case :

If we put n = 1 in our theorem it reduces to the unsymmetrical formula of Hardy and Titchmarsh [1].

In the end I wish to express my thahks to the referee for his suggestions to revise the paper.

REFERENCES

[1] Hardy G. H. & Titchmarsh, E.C. : Proc. London Math. Sco. Ser. II 35 (1933), P. 116.

[2] Saxena, V. P. & Agrawal, B.M. : The Mathematics Student, XXXVII (1969), p. 71.

[3] Fox, C., : Proc. Camb. Phil. Soc., 61 (1965), p. 457.

[4] Titchmarsh, E.C. : Theory of Fourier Integrals', Oxford (1948).

[5] Hardy, G. H. : Proc. London Math. Soc. Ser. II, 23 (1924), p. lxi.

[6] Guinand, A. P. : Quart. J. Math. Oxford Ser. II, 1 (1950), p. 191.