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FRACTIONAL INTEGRATION AND INTEGRAL REPRESENTATIONS OF CERTAIN GENERALIZED HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

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1. INTRODUCTION :

Recently, Joshi [1] has invoked the theory of fractional integration in order to deduce the Eulerian integral representations of hypergeometric functions of three variables viz. H_A, H_B and H_C defined by Srivastava [3, 4, 5, 6, 7].

In the present paper, we shall use the same techniques to derive the integral representations for the multiple hypergeometric functions $F_A^{(n)}, F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ defined by Lauricella [2], and also for their confluent forms $\phi_2^{(n)}$ and $\psi_2^{(n)}$.

We recall that the rule for fractional integration by parts is given by

$$(1.1)\quad \int_a^b U \frac{\partial^{\nu} V}{\partial (b-x)^{\nu}} dx = \int_a^b V \frac{\partial^{\nu} U}{\partial (x-a)^{\nu}} dx.$$

If Re $(\nu) \angle 0$, the fractional derivatives occurring in (1.1) are defined by the following integrals :

$$(1.2)\quad \frac{\partial^{\nu} U}{\partial (x-a)^{\nu}} = \frac{1}{\Gamma(-\nu)} \int_a^x (x-y)^{-\nu-1} U(y)\, dy,$$

$$(1.3)\quad \frac{\partial^{\nu} V}{\partial (b-x)^{\nu}} = \frac{1}{\Gamma(-\nu)} \int_{x}^{b} (y-x)^{-\nu-1} V(y)\, dy.$$

If Re $(\nu) > 0$ and U, V are expressible in terms of the series

$$U = \sum A_r (x-a)^{\rho + r - 1}, \quad V = \sum B_s (b-x)^{\rho + s - 1},$$

then the fractional derivatives are obtained by differentiating the above series term by term, with the help of the formula

$$(1.4)\quad \frac{\partial^{\nu} W^{\mu - 1}}{\partial W^{\nu}} = \frac{\Gamma(\mu)}{\Gamma(\mu-\nu)} W^{\mu - \nu - 1},$$

provided $\mu \neq \nu$.

2. THE INTEGRAL REPRESENTATIONS.

In this section, we derive the integral representations for the functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, $F_D^{(n)}$, $\phi_2^{(n)}$ and $\psi_2^{(n)}$.

Consider

$$\frac{\partial^{\beta_1 - \nu_1 + \beta_2 - \nu_2 + \dots + \beta_n - \nu_n}}{\partial x_1^{\beta_1 - \nu_1} \partial x_2^{\beta_2 - \nu_2} \dots \partial x_n^{\beta_n - \nu_n}} \Big\{ x_1^{\beta_1 - 1} x_2^{\beta_2 - 1} \dots x_n^{\beta_n - 1}$$

$$F_A^{(n)} [\alpha, \nu_1, \nu_2, \dots, \nu_n ; \gamma_1, \gamma_2, \dots, \gamma_n ; x_1, x_2, \dots, x_n] \Big\}$$

$$= \sum_{m_1, m_2, \dots, m_n = 0}^{\infty} \frac{(\alpha)_{m_1 + m_2 + \dots + m_n} (\nu_1)_{m_1} (\nu_2)_{m_2} \dots (\nu_n)_{m_n}}{(\gamma_1)_{m_1} (\gamma_2)_{m_2} \dots (\gamma_n)_{m_n}\; m_1!\, m_2! \dots m_n!}$$

$$\frac{\partial^{\beta_1 - \nu_1 + \beta_2 - \nu_2 + \dots + \beta_n - \nu_n}}{\partial x_1^{\beta_1 - \nu_1} \partial x_2^{\beta_2 - \nu_2} \dots \partial x_n^{\beta_n - \nu_n}} \Big\{ x_1^{\beta_1 + m_1 - 1} x_2^{\beta_2 + m_2 - 1} \dots x_n^{\beta_n + m_n - 1} \Big\}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\beta_j)\, (x_j)^{\nu_j - 1}}{\Gamma(\nu_j)} F_A^{(n)} \Big[\alpha, \beta_1, \beta_2, \dots, \beta_n ; \gamma_1, \gamma_2, \dots, \gamma_n ; x_1, x_2, \dots, x_n \Big]$$

Hence by using the relation (1.2), we have

$$F_A^{(n)} [\alpha, \beta_1, \beta_2,..., \beta_n ; \gamma_1, \gamma_2,..., \gamma_n ; x_1, x_2,..., x_n]$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\nu_j)\,(x_j)^{-\nu_j+1}}{\Gamma(\beta_j)\,\Gamma(\nu_j-\beta_j)} \int_0^{x_1} ...... \int_0^{x_n} (x_1 - y_1)^{\nu_1 - \beta_1 - 1}\, y_1^{\beta_1 - 1} \;......\; (x_n - y_n)^{\nu_n - \beta_n - 1}\, y_n^{\beta_n - 1}$$

$$\times F_A^{(n)} [\alpha, \nu_1, \nu_2,...,\nu_n ; \gamma_1, \gamma_2,...., \gamma_n ; y_1, y_2,..., y_n]\, dy_1\, dy_2...dy_n$$

Now putting every $y_j = x_j t_j$, $j = 1, 2, 3,......, n$, we establish

$$(2.1)\quad F_A^{(n)} [\alpha, \beta_1, \beta_2,..., \beta_n ; \gamma_1, \gamma_2,...., \gamma_n ; x_1, x_2,..., x_n]$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\nu_j)}{\Gamma(\beta_j)\,\Gamma(\nu_j-\beta_j)} \int_0^1 ... \int_0^1 \prod_{i=1}^{n} (t_i)^{\beta_i-1} (1-t_i)^{\nu_i-\beta_i-1}$$

$$\times F_A^{(n)} \Big[\alpha, \nu_1, \nu_2,...,\nu_n ; \gamma_1, \gamma_2,......, \gamma_n ; x_1t_1, x_2t_2,..., x_nt_n \Big] dt_1\, dt_2...dt_n,$$

provided $o < \text{Re}(\beta_i) < \text{Re}(\nu_i)$, $i=1, 2, 3,......, n$ and

$$\sum_{j=1}^{n} |x_j| > 1 .$$

For brevity we use the operator Ω given by

$$(2.2)\quad \Omega_{\nu_1, \nu_1,......, \nu_n}^{\beta_1, \beta_2,......, \beta_n} \{\ \}$$

$$= \prod_{j=1}^{n} \frac{\Gamma(\nu_j)}{\Gamma(\beta_j)\,\Gamma(\nu_j-\beta_j)} \int_0^1 ... \int_0^1 \prod_{i=1}^{n} (t_i)^{\beta_i-1} (1-t_i)^{\nu_i-\beta_i-1} \{\ \} dt_1 dt_2...dt_n,$$

where $o \angle \text{Re}(\beta_i) \angle \text{Re}(\nu_i)$, $i=1, 2, 3,......, n$.

Hence the relation (2.1) can be written as

$$(2.3)\quad \Omega \left\{ F_A^{(n)} [\alpha, \nu_1, \nu_2,..., \nu_n ; \gamma_1, \gamma_2,......, \gamma_n ; x_1t_1, x_2t_2,...x_nt_n] \right\}$$

$$= F_A^{(n)} \, [\alpha, \beta_1, \beta_2, \ldots\ldots, \beta_n; \gamma_1, \gamma_2 \ldots\ldots, \gamma_n ; x_1, x_2, \ldots\ldots, x_n],$$

provided $0 \angle \text{Re}(\beta_i) \angle \text{Re}(\nu_i)$, $i = 1, 2, 3, \ldots\ldots, n$ and $\sum_{j=1}^{n} | x_j | < 1$.

It is also easy to prove that

(2.4) $\Omega \{1\} = 1.$

Applying the same techniques, we also obtain the following operational relations :—

(2.5) $$\Omega \left\{ F_B^{(n)} \, [u_1, u_2, \ldots\ldots, u_n, \nu_1, \nu_2, \ldots\ldots, \nu_n; \gamma; x_1t_1, x_2t_2, \ldots\ldots, x_nt_n] \right\}$$

$$= F_B^{(n)} \, [u_1, u_2, \ldots\ldots, u_n, \beta_1, \beta_2, \ldots\ldots, \beta_n; \gamma; x_1, x_2, \ldots\ldots, x_n],$$

provided $0 \angle \text{Re}(\beta_i) \angle \text{Re}(\nu_i)$, and $| x_i | \angle 1$, $i=1, 2, 3, \ldots\ldots, n$.

(2.6) $$\Omega \left\{ F_C^{(n)} \, [\alpha, \gamma; \beta_1, \beta_2, \ldots\ldots, \beta_n; x_1t_1, x_2t_2, \ldots\ldots, x_nt_n] \right\}$$

$$= F_C^{(n)} \, [\alpha, \gamma; \nu_1, \nu_2, \ldots\ldots, \nu_n; x_1, x_2, \ldots\ldots, x_n],$$

valid if $0 \angle \text{Re}(\beta_i) \angle \text{Re}(\nu_i)$, $i=1, 2, 3, \ldots\ldots, n$ and $\sum_{j=1}^{n} | x_j |^{1/2} \angle 1$.

(2.7) $$\Omega \left\{ F_D^{(n)} \, [\alpha, \nu_1, \nu_2, \ldots\ldots, \nu_n; \gamma; x_1t_1, x_2t_2, \ldots\ldots, x_nt_n] \right\}$$

$$= F_D^{(n)} \, [\alpha, \beta_1, \beta_2, \ldots\ldots, \beta_n; \gamma; x_1, x_2, \ldots\ldots, x_n],$$

provided that the conditions of (2.3) hold true.

(2.8) $$\Omega \left\{ \phi_2^{(n)} \, [\nu_1, \nu_2, \ldots\ldots, \nu_n; \gamma; x_1t_1, x_2t_2, \ldots\ldots, x_nt_n] \right\}$$

$$= \phi_2^{(n)} [\beta_1, \beta_2, \ldots, \beta_n; \gamma; x_1, x_2, \ldots, x_n],$$

where $o < \text{Re}(\beta_i) < \text{Re}(\nu_i)$, $i=1, 2, 3, \ldots, n$.

$$(2.9) \quad \Omega \left\{ \psi_2^{(n)} [\alpha; \beta_1, \beta_2, \ldots, \beta_n; x_1 t_1, x_2 t_2, \ldots, x_n t_n] \right\}$$

$$= \psi_2^{(n)} [\alpha; \nu_1, \nu_2, \ldots, \nu_n; x_1, x_2, \ldots, x_n],$$

valid under the conditions $o < \text{Re}(\beta_i) < \text{Re}(\nu_i)$, $i=1, 2, 3, \ldots, n$.

$$(2.10) \quad \Omega \left\{ F_A^{(n)} [\nu_1, \alpha_1, \alpha_2, \ldots, \alpha_n; \gamma_1, \beta_2, \beta_3, \ldots, \beta_n; x_1 t_1, x_1 x_2 t_1 t_2, \ldots, x_1 x_n t_1 t_n] \right\}$$

$$= F_A^{(n)} [\beta_1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n; \gamma_1, \nu_2, \nu_3, \ldots, \nu_n; x_1, x_1 x_2, \ldots, x_1 x_n],$$

provided $|x_1| + |x_1 x_2| + \ldots + |x_1 x_n| < 1$ and $o < \text{Re}(\beta_i) < \text{Re}(\nu_i)$, $i=1, 2, 3, \ldots, n$.

$$(2.11) \quad \Omega \left\{ F_B^{(n)} [\alpha_1, \nu_2, \nu_3, \ldots, \nu_n \gamma_1, \gamma_2, \ldots, \gamma_n; \beta_1; x_1 t_1, x_1 x_2 t_1 t_2, \ldots, x_1 x_n t_1 t_n] \right\}$$

$$= F_B^{(n)} [\alpha_1, \beta_2, \beta_3, \ldots \beta_n, \gamma_1, \gamma_2, \ldots \gamma_n; \nu_1; x_1, x_1 x_2, \ldots, x_1 x_n],$$

where $o < \text{Re}(\beta_i) < \text{Re}(\nu_i)$, $i=1, 2, 3, \ldots, n$ and $|x_1| < 1$, all $|x_1 x_j| < 1$, $j=2, 3, 4, \ldots, n$.

$$(2.12) \quad \Omega \left\{ F_C^{(n)} [\nu_1, \alpha; \gamma_1, \beta_2, \beta_3, \ldots, \beta_n; x_1 t_1, x_1 x_2 t_1 t_2, \ldots, x_1 x_n t_1 t_n] \right\}$$

$$= F_C^{(n)} [\beta_1, \alpha; \gamma_1, \nu_2, \nu_3, \ldots, \nu_n; x_1, x_1 x_2, \ldots, x_1 x_n],$$

provided $o < \text{Re}(\beta_i) < \text{Re}(\nu_i)$, $i=1, 2, 3, \ldots, n$ and $|x_1|^{\frac{1}{2}} + \sum_{j=2}^{n} |x_i x_j|^{\frac{1}{2}} < 1$.

$$(2.13) \quad \Omega \left\{ F_D^{(n)} [\nu_1, \gamma_1, \nu_2, \nu_3, \ldots, \nu_n; \gamma; x_1 t_1, x_1 x_2 t_1 t_2, \ldots, x_1 x_n t_1 t_n] \right\}$$

$$= \phi_2^{(n)} [\beta_1, \beta_2, \ldots, \beta_n; \gamma; x_1, x_2, \ldots, x_n],$$

where $o < \mathrm{Re}(\beta_i) < \mathrm{Re}(\nu_i)$, $i=1, 2, 3, \ldots, n$.

$$(2.9)\ \Omega \left\{ \psi_2^{(n)} [a; \beta_1, \beta_2, \ldots, \beta_n; x_1 t_1, x_2 t_2, \ldots, x_n t_n] \right\}$$

$$= \psi_2^{(n)} [a; \nu_1, \nu_2, \ldots, \nu_n; x_1, x_2, \ldots, x_n],$$

valid under the conditions $o < \mathrm{Re}(\beta_i) < \mathrm{Re}(\nu_i)$, $i=1, 2, 3, \ldots, n$.

$$(2.10)\ \Omega \left\{ F_A^{(n)} [\nu_1, a_1, a_2, \ldots, a_n; \gamma_1, \beta_2, \beta_3, \ldots, \beta_n; x_1 t_1, x_1 x_2 t_1 t_2, \ldots, x_1 x_n t_1 t_n] \right\}$$

$$= F_A^{(n)} [\beta_1, a_1, a_2, a_3, \ldots, a_n; \gamma_1, \nu_2, \nu_3, \ldots, \nu_n; x_1, x_1 x_2, \ldots, x_1 x_n],$$

provided $|x_1| + |x_1 x_2| + \ldots + |x_1 x_n| < 1$ and $o < \mathrm{Re}(\beta_i) < \mathrm{Re}(\nu_i)$, $i=1, 2, 3, \ldots, n$.

$$(2.11)\ \Omega \left\{ F_B^{(n)} [a_1, \nu_2, \nu_3, \ldots, \nu_n\ \gamma_1, \gamma_2, \ldots, \gamma_n;\ \beta_1;\ x_1 t_1,\ x_1 x_2 t_1 t_2, \ldots, x_1 x_n t_1 t_n] \right\}$$

$$= F_B^{(n)} [a_1, \beta_2, \beta_3, \ldots \beta_n, \gamma_1, \gamma_2, \ldots \gamma_n; \nu_1; x_1, x_1 x_2, \ldots, x_1 x_n],$$

where $o < \mathrm{Re}(\beta_i) < \mathrm{Re}(\nu_i)$, $i=1, 2, 3, \ldots, n$ and $|x_1| < 1$, all $|x_1 x_j| < 1$, $j=2, 3, 4, \ldots, n$.

$$(2.12)\ \Omega \left\{ F_C^{(n)} [\nu_1, a; \gamma_1, \beta_2, \beta_3, \ldots, \beta_n; x_1 t_1, x_1 x_2 t_1 t_2, \ldots, x_1 x_n t_1 t_n] \right\}$$

$$= F_C^{(n)} [\beta_1, a; \gamma_1, \nu_2, \nu_3, \ldots, \nu_n; x_1, x_1 x_2, \ldots, x_1 x_n],$$

provided $o < \mathrm{Re}(\beta_i) < \mathrm{Re}(\nu_i)$, $i=1, 2, 3, \ldots, n$ and $|x_1|^{\frac{1}{2}} + \sum_{j=2}^{n} |x_i x_j|^{\frac{1}{2}} < 1$.

$$(2.13)\ \Omega \left\{ F_D^{(n)} [\nu_1, \gamma_1, \nu_2, \nu_3, \ldots, \nu_n; \gamma; x_1 t_1, x_1 x_2 t_1 t_2, \ldots, x_1 x_n t_1 t_n] \right\}$$

$$= F_D^{(n)} [\beta_1, \gamma_1, \beta_2, \beta_3, \ldots, \beta_n; \gamma; x_1, x_1x_2, \ldots, x_1x_n],$$

valid if all the conditions of (2.10) hold true.

$$(2.14) \quad \Omega \left\{ \phi_2^{(n)} [\alpha_1, \nu_2, \nu_3, \ldots, \nu_n; \beta_1; x_1t_1, x_1x_2t_1t_2, \ldots, x_1x_nt_1t_n] \right\}$$

$$= \phi_2^{(n)} [\alpha_1, \beta_2, \beta_3, \ldots, \beta_n; \nu_1; x_1, x_1x_2, \ldots, x_1x_n],$$

provided $o \angle Re(\beta_i) \angle Re(\nu_i)$, $i=1, 2, 3, \ldots, n$.

$$(2.15) \quad \Omega \left\{ \psi_2^{(n)} [\nu_1; \alpha_1; \beta_2, \beta_3, \ldots, \beta_n; x_1t_1, x_1x_2\, t_1t_2, \ldots, x_1x_nt_1t_n] \right\}$$

$$= \psi_2^{(n)} [\beta_1; \alpha_1, \nu_2, \nu_3, \ldots, \nu_n; x_1, x_1x_2, \ldots, x_1\, x_n]$$

under the conditions $o \angle Re(\beta_i) \angle Re(\nu_i)$, $i=1, 2, 3, \ldots, n$.

3. FURTHER EXTENSIONS.

In the present section we derive extensions of the results obtained in the preceding section to hold for the generalized hypergeometric functions of n-variables, defined by

$$\tilde{F}(x_1, x_2, \ldots, x_n) = F\left[\begin{array}{c|c|c} \begin{array}{c} 2 \\ o \\ \vdots \\ o \\ n \\ 1 \\ o \\ \vdots \\ o \\ n \end{array} & \begin{array}{c} \nu_1; \lambda_1 \\ \text{-------} \\ \\ \text{-------} \\ \alpha_1; \alpha_2, \lambda_2; \ldots; \alpha_n, \lambda_n \\ \mu_1 \\ \text{-------} \\ \\ \text{-------} \\ \gamma_1; \beta_2, \mu_2; \beta_3, \mu_3; \ldots; \beta_n, \mu_n \end{array} & x_1, x_2, \ldots, x_n \end{array}\right]$$

$$= \sum_{m_1, m_2, \ldots, m_n = o}^{\infty} \frac{(\nu_1)_{m_1+m_2+\ldots+m_n} (\lambda_1)_{m_1+m_2+\ldots+m_n} (\lambda_2)_{m_2} \cdots (\lambda_n)_{m_n}}{(\mu_1)_{m_1+m_2+\ldots+m_n} (\mu_2)_{m_2} \cdots (\mu_n)_{m_n} \; (\gamma_1)_{m_1} \; (\beta_2)_{m_2} \cdots}$$

$$\frac{(\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n} \; x_1^{m_1} \cdots x_n^{m_n}}{(\beta_n)_{m_n} \; m_1!\, m_2! \ldots m_n!}$$

From the above definition it follows that

$$\frac{\partial^{\mu_1-\lambda_1+\dots+\mu_n-\lambda_n}}{\partial t_1^{\mu_1-\lambda_1}\dots\partial t_n^{\mu_n-\lambda_n}}\left\{t_1^{\mu_1-1}\, t_2^{\mu_2-1}\dots t_n^{\mu_n-1}\,\tilde{F}(x_1t_1, x_1x_2t_1t_2,\dots, x_1x_nt_1t_n)\right\}$$

$$= \prod_{j=1}^{n}\frac{\Gamma(\mu_j)\,(t_j)^{\gamma_j-1}}{\Gamma(\lambda_j)}\, F_A^{(n)}[\nu_1, a_1, a_2,\dots,a_n; \gamma_1, \beta_2, \beta_3,\dots, \beta_n; x_1t_1, x_1x_2t_1t_2,\dots,x_1x_nt_1t_n].$$

Using the relation (2.10), we have

$$F_A^{(n)}[\beta_1, a_1, a_2,\dots, a_n; \gamma_1, \nu_2, \nu_3,\dots, \nu_n; x_1, x_1x_2,\dots, x_1x_n]$$

$$= \prod_{j=1}^{n}\frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\,\Gamma(\mu_j)\,\Gamma(\nu_j-\beta_j)}\int_0^1\!\!\dots\!\int_0^1 \prod_{i=1}^{n}(t_i)^{\beta_i-\lambda_i}(1-t_i)^{\nu_i-\beta_i-1}$$
$$\frac{\partial^{\mu_1-\lambda_1+\dots+\mu_n-\lambda_n}}{\partial t_1^{\mu_1-\lambda_1}\dots\partial t_n^{\mu_n-\lambda_n}}\left\{t_1^{\mu_1-1}\, t_2^{\mu_2-1}\dots t_n^{\mu_n-1}\,\tilde{F}(x_1t_1, x_1x_2t_1t_2,\dots, x_1x_nt_1t_n)\right\}dt_1dt_2\dots dt_n,$$

so that an appeal to (1.1) gives us

$$F_A^{(n)}[\beta_1, a_1, a_2,\dots, a_n; \gamma_1, \nu_2, \nu_3,\dots, \nu_n; x_1, x_1x_2,\dots, x_1x_n]$$

$$= \prod_{j=1}^{n}\frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\,\Gamma(\mu_j)\,\Gamma(\nu_j-\beta_j)}\int_0^1\!\!\dots\!\int_0^1 t_1^{\mu_1-1}\, t_2^{\mu_2-1}\dots\dots t_n^{\mu_n-1}\,\tilde{F}(x_1t_1, x_1x_2t_1t_2,\dots, x_1x_nt_1t_n)$$
$$\times\frac{\partial^{\mu_1-\lambda_1+\mu_2-\lambda_2+\dots+\mu_n-\lambda_n}}{\partial(1-t_1)^{\mu_1-\lambda_1}\dots\partial(1-t_n)^{\mu_n-\lambda_n}}\left\{\prod_{i=1}^{n}(t_i)^{\beta_i-\lambda_i}(1-t_i)^{\nu_i-\beta_i-1}\right\}dt_1dt_2\dots dt_n$$

$$= \prod_{j=1}^{n}\frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\,\Gamma(\mu_j)\,\Gamma(\nu_j+\lambda_j-\mu_j-\beta_j)}\int_0^1\!\!\dots\!\int_0^1 \prod_{i=1}^{n}(t_i)^{\mu_i-1}(1-t_i)^{\nu_i+\lambda_i-\mu_i-\beta_i-1}$$

$$\times \; {}_2F_1 \, [\nu_i - \beta_i, \lambda_i - \beta_i; \nu_i + \lambda_i - \mu_i - \beta_i; (1-t_i)] \, \tilde{F} \, (x_1 t_1, x_1 x_2 t_1 t_2, \ldots, x_1 x_n t_1 t_n) \, dt_1 dt_2 \ldots dt_n.$$

For brevity, we define an operator R by the relation

$$(3.1) \; \mathbf{R} \left\{ \; \right\} = \prod_{j=1}^{n} \frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\,\Gamma(\mu_j)\,\Gamma(\nu_j+\lambda_j-\beta_j-\mu_j)} \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} (t_i)^{\mu_i - 1}$$

$$\times \; (1-t_i)^{\nu_i+\lambda_i-\mu_i-\beta_i-1} \; {}_2F_1 \, [\nu_i - \beta_i, \lambda_i - \beta_i; \nu_i+\lambda_i-\mu_i-\beta_i; (1-t_i)] \left\{ \; \right\} dt_1 \ldots dt_n \, .$$

We then have

$$(3.2) \; \mathbf{R} \left\{ F \left[\begin{array}{c|c} 2 & \nu_1 ; \lambda_1 \\ 0 & \text{------} \\ \vdots & \\ 0 & \text{------} \\ n & \alpha_1 ; \alpha_2, \lambda_2 ; \ldots ; \alpha_n, \lambda_n \\ 1 & \mu_1 \\ 0 & \text{------} \\ \vdots & \\ 0 & \text{------} \\ n & \gamma_1 ; \beta_2, \mu_2 ; \ldots ; \beta_n, \mu_n \end{array} \middle| \; x_1 t_1, x_1 x_2 t_1 t_2, \ldots, x_1 x_n t_1 t_n \right] \right\}$$

$$= F_A^{(n)} \, [\beta_1, \alpha_1, \alpha_2, \ldots, \alpha_n; \gamma_1, \nu_2, \nu_3, \ldots, \nu_n; x_1, x_1 x_2, \ldots, x_1 x_n],$$

provided $0 \angle \operatorname{Re}(\mu_j) \angle \operatorname{Re}(\nu_j+\lambda_j-\beta_j)$, $0 \angle \operatorname{Re}(\beta_j) \angle \operatorname{Re}(\nu_j)$, $j=1, 2, 3, \ldots, n$ and $|x_1| + |x_1 x_2| + \ldots + |x_1 x_n| \angle 1$.

Similarly we obtain the following results :—

$$(3.3) \; \mathbf{R} \left\{ F \left[\begin{array}{c|c} 1 & \lambda_1 \\ 0 & \text{------} \\ \vdots & \\ 0 & \text{------} \\ n & \gamma_1, \alpha_1 ; \gamma_2, \nu_2, \lambda_2 ; \ldots ; \gamma_n, \nu_n, \lambda_n \\ 2 & \mu_1 ; \beta_1 \\ 0 & \text{------} \\ \vdots & \\ 0 & \text{------} \\ n\text{-}1 & — ; \mu_2 ; \mu_3 ; \ldots ; \mu_n \end{array} \middle| \; x_1 t_1, x_1 x_2 t_1 t_2, \ldots, x_1 x_n t_1 t_n \right] \right\}$$

$$= F_B^{(n)}[\alpha_1, \beta_2, \beta_3,\ldots, \beta_n; \gamma_1, \gamma_2,\ldots, \gamma_n; \nu_1; x_1\, x_2,\ldots, x_1\, x_n],$$

valid under the conditions $o < \mathrm{Re}\,(\mu_j) < \mathrm{Re}\,(\nu_j+\lambda_j-\beta_j)$, $o < \mathrm{Re}\,(\beta_j) < \mathrm{Re}\,(\nu_j)$, $j=1,2,3,\ldots, n$, and $|x_1| < 1$, all $|x_1\, x_i| < 1$, $i = 2, 3, 4, \ldots, n$.

$$(3.4)\quad \mathbf{R}\left\{ F\left[\begin{array}{c|c|c} 3 & \lambda_1;\ \nu_1;\ \alpha & \\ o & \text{---------} & \\ \vdots & & \\ o & \text{---------} & \\ n\text{-}1 & -;\ \lambda_2;\ \lambda_3;\ldots;\ \lambda_n & x_1t_1, x_1x_2t_1t_2,\ldots, \\ 1 & \mu_1 & x_1x_nt_1t_n \\ o & \text{---------} & \\ \vdots & & \\ o & \text{---------} & \\ n & \gamma_1;\ \beta_2,\ \mu_2;\ \beta_3,\ \mu_3;\ldots;\ \beta_n,\ \mu_n & \end{array}\right]\right\}$$

$$= F_C^{(n)}\,[\beta_1, \alpha; \gamma_1, \nu_2, \nu_3,\ldots, \nu_n; x_1, x_1x_2,\ldots, x_1\, x_n],$$

provided $o < \mathrm{Re}\,(\mu_j) < \mathrm{Re}\,(\nu_j+\lambda_j-\beta_j)$, $o < \mathrm{Re}\,(\beta_j) < \mathrm{Re}\,(\nu_j)$,

$j = 1, 2, 3,\ldots, n$ and $|\, x_1 \,|^{\frac{1}{2}} + \sum\limits_{i=2}^{n} |\, x_1\, x_i \,|^{\frac{1}{2}} < 1.$

$$(3.5)\quad \mathbf{R}\left\{ F\left[\begin{array}{c|c|c} 2 & \lambda_1;\ \nu_1 & \\ o & \text{---------} & \\ \vdots & & \\ o & \text{---------} & \\ n & \gamma_1;\ \lambda_2,\ \nu_2;\ldots;\ \lambda_n,\ \nu_n & x_1t_1, x_1x_2t_1t_2,\ldots, x_1x_nt_1t_n \\ 2 & \gamma;\ \mu_1 & \\ o & \text{---------} & \\ \vdots & & \\ o & \text{---------} & \\ n\text{-}1 & -;\ \mu_2;\ \mu_3;\ldots;\ \mu_n & \end{array}\right]\right\}$$

$$= F_D^{(n)}\,[\beta_1, \gamma_1, \beta_2, \beta_3,\ldots, \beta_n; \gamma; x_1, x_1\, x_2,\ldots, x_1\, x_n],$$

valid if all the conditions of (3.2) hold true.

$$(3.6)\quad \mathbf{R}\left\{ F\left[\begin{array}{c|c|c} 1 & \lambda_1 & \\ o & \text{---------} & \\ \vdots & & \\ o & \text{---------} & \\ n & \alpha_1;\ \lambda_2,\ \nu_2;\ldots;\ \lambda_n,\ \nu_n & x_1t_1, x_1x_2t_1t_2,\ldots, x_1x_nt_1t_n \\ 2 & \mu_1;\ \beta_1 & \\ o & \text{---------} & \\ \vdots & & \\ o & \text{---------} & \\ n\text{-}1 & -;\ \mu_2;\ \mu_3;\ldots;\ \mu_n & \end{array}\right]\right\}$$

$$= \phi_2^{(n)}\left[\alpha_1, \beta_2, \beta_3, \ldots, \beta_n; \nu_1; x_1, x_1 x_2, x_1 x_3, \ldots, x_1 x_n\right],$$

provided $o < \mathrm{Re}(\mu_i) < \mathrm{Re}(\nu_i + \lambda_i - \beta_i)$, $o < \mathrm{Re}<(\beta_i) < \mathrm{Re}(\nu_i)$, $i = 1, 2, 3, \ldots, n$.

$$(3.7)\quad R\left\{ F\left[\begin{array}{c|c|c} 2 & \lambda_1; \nu_1 & \\ o & \text{------} & \\ \vdots & & \\ o & \text{------} & \\ n\text{-}1 & -; \lambda_2; \lambda_3; \ldots; \lambda_n & x_1 t_1, x_1 x_2 t_1 t_2, \ldots, \\ 1 & \mu_1 & x_1 x_n t_1 t_n \\ o & \text{------} & \\ \vdots & & \\ o & \text{------} & \\ n & \alpha_1; \mu_2, \beta_2; \mu_3, \beta_3; \ldots; \mu_n, \beta_n & \end{array}\right]\right\}$$

$$= \psi_2^{(n)}\left[\beta_1; \alpha_1, \nu_2, \nu_3, \ldots, \nu_n; x_1, x_1 x_2, \ldots, x_1 x_n\right],$$

where $o < \mathrm{Re}(\mu_i) < \mathrm{Re}(\nu_i + \lambda_i - \beta_i)$, $o < \mathrm{Re}(\beta_i) < \mathrm{Re}(\nu_i)$, $i = 1, 2, 3, \ldots, n$.

$$(3.8)\quad R\left\{ F\left[\begin{array}{c|c|c} 1 & \alpha & \\ o & \text{------} & \\ \vdots & & \\ o & \text{------} & \\ n & \nu_1, \lambda_1; \nu_2, \lambda_2; \ldots; \nu_n, \lambda_n & x_1 t_1, x_2 t_2, \ldots, x_n t_n \\ o & \text{------} & \\ \vdots & & \\ o & \text{------} & \\ n & \mu_1, \gamma_1; \mu_2, \gamma_2; \ldots; \mu_n, \gamma_n & \end{array}\right]\right\}$$

$$= F_A^{(n)}\left[\alpha, \beta_1, \beta_2, \ldots, \beta_n; \gamma_1, \gamma_2, \ldots, \gamma_n; x_1, x_2, \ldots, x_n\right],$$

provided $0 < \mathrm{Re}(\mu_i) < \mathrm{Re}(\nu_i + \lambda_i - \beta_i)$, $0 < \mathrm{Re}(\beta_i) < \mathrm{Re}(\nu_i)$, $i = 1, 2, 3, \ldots\ldots\ldots n$

and $\sum_{j=1}^{n} |x_j| < 1$.

$$(3.9)\quad R\left\{ F\left[\begin{array}{c|c|c} o & \text{------} & \\ \vdots & & \\ o & \text{------} & \\ n & u_1, \nu_1, \lambda_1; u_2, \nu_2, \lambda_2; \ldots; u_n, \nu_n, \lambda_n & x_1 t_1, x_2 t_2, \ldots, x_n t_n \\ 1 & \gamma & \\ o & \text{------} & \\ \vdots & & \\ o & \text{------} & \\ n & \mu_1; \mu_2; \ldots; \mu_n & \end{array}\right]\right\}$$

$$= F_B^{(n)}\left[u_1, u_2, \ldots, u_n, \beta_1, \beta_2, \ldots, \beta_n; \gamma; x_1, x_2, \ldots, x_n\right],$$

where $0 < \mathrm{Re}\,(\mu_i) < \mathrm{Re}\,(\nu_i+\lambda_i-\beta_i)$, $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i)$ and all $|x_i| < 1$, $i=1, 2, 3,\ldots, n$.

$$(3.10)\quad \mathbf{R}\left\{ F\left[\begin{array}{c|c|c} 2 & \alpha;\ \gamma & \\ 0 & ------------ & \\ \vdots & & \\ 0 & ------------ & \\ n & \lambda_1;\ \lambda_2;\ldots;\ \lambda_n & x_1t_1,\ x_2t_2,\ldots,\ x_nt_n \\ 0 & ------------ & \\ \vdots & & \\ 0 & ------------ & \\ n & \mu_1,\ \beta_1;\ \mu_2,\ \beta_2;\ldots;\ \mu_n,\ \beta_n & \end{array}\right]\right\}$$

$$= F_C^{(n)}[\alpha, \gamma;\ \nu_1, \nu_2,\ldots, \nu_n;\ x_1, x_2,\ldots, x_n],$$

provided $0 < \mathrm{Re}\,(\mu_i) < \mathrm{Re}\,(\nu_i+\lambda_i-\beta_i)$, $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i)$,

$i=1, 2, 3,\ldots, n$ and $\sum_{j=1}^{n} |x_j|^{\frac{1}{2}} < 1.$

$$(3.11)\quad \mathbf{R}\left\{ F\left[\begin{array}{c|c|c} 1 & \alpha & \\ 0 & ---------- & \\ \vdots & & \\ 0 & ---------- & \\ n & \nu_1,\ \lambda_1;\ \nu_2,\ \lambda_2;\ldots;\nu_n,\ \lambda_n & \\ 1 & \gamma & x_1t_1,\ x_2t_2,\ldots,\ x_nt_n \\ 0 & ---------- & \\ \vdots & & \\ 0 & ---------- & \\ n & \mu_1;\ \mu_2;\ldots;\ \mu_n & \end{array}\right]\right\}$$

$$= F_D^{(n)}[\alpha, \beta_1, \beta_2,\ldots, \beta_n;\ \gamma;\ x_1, x_2,\ldots, x_n],$$

provided that all the conditions of (3.8) hold true.

$$(3.12)\quad \mathbf{R}\left\{ F\left[\begin{array}{c|c|c} 0 & ---------- & \\ \vdots & & \\ 0 & ---------- & \\ n & \nu_1,\ \lambda_1;\ \nu_2,\ \lambda_2;\ldots;\ \nu_n,\ \lambda_n & \\ 1 & \gamma & x_1t_1,\ x_2t_2,\ldots,\ x_nt_n \\ 0 & ---------- & \\ \vdots & & \\ 0 & ---------- & \\ n & \mu_1;\ \mu_2;\ \mu_3;\ldots;\ \mu_n & \end{array}\right]\right\}$$

$$= \phi_2^{(n)}\,[\beta_1, \beta_2,\ldots, \beta_n;\ \gamma;\ x_1, x_2,\ldots, x_n],$$

valid if $0 < \text{Re}(\mu_i) < \text{Re}(\nu_i+\lambda_i-\beta_i)$, $0 < \text{Re}(\beta_i) < \text{Re}(\nu_i)$, $i=1, 2, 3,\ldots, n$.

$$(3.13)\quad R\left\{ F\left[\begin{array}{c|c} 1 & \alpha \\ 0 & \text{-----------} \\ \vdots & \\ 0 & \text{-----------} \\ n & \lambda_1; \lambda_2;\ldots; \lambda_n \\ 0 & \text{-----------} \\ \vdots & \\ 0 & \text{-----------} \\ n & \mu_1, \beta_1; \mu_2, \beta_2;\ldots; \mu_n, \beta_n \end{array} \; x_1 t_1, x_2 t_2,\ldots, x_n t_n \right] \right\}$$

$$= \psi_2^{(n)}[\alpha; \nu_1, \nu_2,\ldots, \nu_n; x_1, x_2,\ldots, x_n],$$

valid if all the conditions of (3.12) hold true.

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