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THERMAL STRESS IN AN ELASTIC SPHERE CONTAINING A PENNY-SHAPED CRACK*

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1. INTRODUCTION

In a recent paper [1] we have considered the problem of finding the distribution of thermal stresses in an elastic sphere containing a panny-shaped crack situated in a diametral plane. In [1] we have considered the case when the temperature on the surfaces of the crack is prescribed. Here we shall study the problem of finding thermal stresses when the flux of heat across the surfaces of crack is prescribed.

FORMULATION OF THE PROBLEM

We shall consider the temperature and displacement fields in a perfectly elastic sphere conducting heat. With regards to both the thermal and mechanical properties the sphere will be assumed to be isotropic and homogeneous. In the problems we shall consider, it will be assumed that there is symmetry with respect to the line $\theta=0$. The position of a typical point may be expressed in terms of spherical polar co-ordinates (r, θ, φ). For a symmetrical deformation the displacement vector $\underline{U}$ may be taken to have the components $\{U_r(r, \theta), U_\theta(r, \theta), 0\}$ and the only non-vanishing components of stress tensor will be σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{\varphi\varphi}$, $\sigma_{r\theta}$. We suppose that the radius of the crack is the unit of length. The diametral plane in which the crack lies is taken to be a co-ordinate plane. The centre of the crack is the

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origin and the straight line through origin and perpendicular to this plane is the $\theta=0$ line.

The problem of finding the distribution of thermal stress in the vicinity of the crack is equivalent to that of finding the state of stress in a hemisphere when its plane boundary is subjected to the thermal and elastic conditions.

$$\sigma_{r\theta}(r, \pi/2)=0, \qquad 0 \leqslant r \leqslant c \tag{1.1}$$

$$\sigma_{\theta\theta}(r, \pi/2)=0, \qquad 0 \leqslant r \leqslant 1 \tag{1.2}$$

$$U_{\theta}(r, \pi/2)=0, \qquad 1 \leqslant r \leqslant c \tag{1.3}$$

where $c >> 1$ is the radius of the sphere. Since the flux of heat across the surface of the crack is prescribed, we have

$$\left[\frac{1}{r}\frac{\delta T}{\delta r}(r, \theta)\right]_{\theta=\frac{\pi}{2}} = f(r), \; 0 \leqslant r \leqslant 1 \tag{1.4}$$

$$= 0 \quad , 1 \leqslant r \leqslant c \tag{1.5}$$

The two problems to be studied are

Problem 1

The curved surface of the sphere is free from shear and is supported in such a way that the radial component of displacement vector vanishes on the surface which is kept at zero temperature. Such a situation will arise physically when the sphere is resting in a spherical hollow in a rigid body of exactly the same radius. These conditions correspond to lubricated contact. The fact that the sphere is held in this way implies that

$$T(c, \theta) = U_r(c, \theta) = \sigma_{r\theta}(c, \theta) = 0 \quad , 0 \leqslant \theta \leqslant \pi/2 \tag{1.6}$$

Problem 2

The surface of the sphere is free from traction and is kept at zero temperature. The boundary conditions in this case are

$$T(c, \theta) = \sigma_{r\theta}(c, \theta) = \sigma_{rr}(c, \theta) = 0 \quad , 0 \leqslant \theta \leqslant \pi/2 \tag{1.7}$$

In sections 2 to 4 the mixed boundary value problems posed by the above equations are reduced to a Fredholm integral equations of the second kind which are solved by iterative process. In section 5, the practically important case, when the flux across the crack surface is constant, is considered. Expressions for quantities of physical interest which are valid for small values of l/c are derived from the iterative solution of the basic equation. In sect tion 6, the values of these quantities for values of l/c nearly equal to unity are derived from the numerical solution of the basic Fredholm integral equations.

2. SOLUTIONS OF THE EQUATIONS OF EQUILIBRIUM FOR THE THERMAL FIELD

A complete solution of the equation of thermoelastic equilibrium, suitable for the problems stated above have been derived in [1]. The expression for temperature, displacement and stress components are

$$T(r, \theta) = \int_0^{\infty} \varphi(\xi)\, e^{-\xi r \cos\theta} \left[J_0 (\xi r \sin\theta) \right] d\xi + + \sum_{n=0}^{\infty} d_{2n}\, r^{2n}\, P_{2n} (\cos\theta) \tag{2.1}$$

$$\begin{aligned} U_r(r, \theta) = {} & \int_0^{\infty} e^{-\xi r \cos\theta} \Big[J_1 (\xi r \sin\theta) \sin\theta \\ & \{A(\xi)(1+\xi r \cos\theta) - (2-2\eta) B(\xi)\} + \\ & + J_0 (\xi r \sin\theta) \cos\theta \{A(\xi)\, \xi r \cos\theta + \\ & + B(\xi)(2-2\eta)\} \Big] d\xi + \sum_{n=0}^{\infty} \Big[\Big\{ a_{2n} (2n+1) \\ & (2n+4\eta-2) + \frac{2(1+\eta)t}{2n+5-4\eta} d_{2n} \Big\} r^{2n+1} + \\ & + 2n\, b_{2n} (\nu)^{2n-1} \Big] P_{2n} (\cos\theta) \end{aligned} \tag{2.2}$$

$$U_\theta\ (r,\theta) = \int_0^\infty e^{-\xi r \cos\theta} \Big[J_1\,(\xi r \sin\theta) \cos\theta$$

$$\Big\{ A(\xi)\ (1+\xi r\cos\theta) - (2-2\eta)\, B(\xi) \Big\} -$$

$$-\ J_0\ (\xi r \sin\theta) \sin\theta \Big\{ A(\xi)\, \xi r \cos\theta\ +$$

$$+\ (2-2\eta)\, B(\xi) \Big\} \Big] d\xi + \sum_{n=1}^{\infty} \Big[\ a_{2n}\ (2n\ +$$

$$+\ 5-4\eta)\, r^{2n+1} + b_{2n}\ r^{2n-1} \Big] \frac{d}{d\theta}\ P_{2n}(\cos\theta) \tag{2.3}$$

$$\frac{\sigma_{rr}(r,\theta)}{2\mu} = \int_0^\infty \frac{e^{-\xi r\cos\theta}}{r} \Big[J_1\,(\xi r \sin\theta) \Big\{ B(\xi)\ (2-2\eta)\ -$$

$$-\ A(\xi)\ (1+\xi r\cos\theta + 2\,\xi^2 r^2 \cos^2\theta) \Big\} \sin\theta\ -$$

$$-\ J_0\,(\xi r \sin\theta)\, \xi r\, (1+\xi r\cos\theta \cos 2\theta) \Big] d\xi\ \ +$$

$$+ \sum_{n=0}^{\infty} \Big[\Big\{ a_{2n}\ (2n+1)\ (4n^2-2n-2\eta-2)\ +$$

$$+\ \frac{(2n-3)\ (1+\eta)\ \ \alpha_t}{2n+5-4\eta}\ d_{2n} \Big\}\ r^{2n} + 2n\, b_{2n}\ (2n-1)\ \ r^{2n-2} \Big]$$

$$P_{2n}\ (\cos\theta) \tag{2.4}$$

$$\frac{\sigma_{\theta\theta}(r,\theta)}{2\mu} = \int_0^\infty \xi\, e^{-\xi r\cos\theta} \Big[\frac{J_1\ (\xi r \sin\theta)\ \cos^2\theta}{\xi r \sin\theta}$$

$$\Big\{ B(\xi)(2-2\eta) - A(\xi)(1+\xi r\cos\theta - 2\xi^2 r^2 \sin^2\theta) \Big\}$$

$$+ J_0 (\xi r \sin\theta) A(\xi)(\xi r \cos\theta \cos 2\theta - 1) \Big\} d\xi -$$

$$-\sum_{n=0}^{\infty} \Big[\Big\{ a_{2n} (4n^2+8n+2\eta+2)(2n+1) +$$

$$+ \frac{(2n+3)(1+\eta)\,\alpha_t}{2n+5-4\eta} \quad d_{2n} \Big\} r^{2n} + 4n^2 \quad b_{2n} \quad r^{2n-2} \Big]$$

$$P_{2n} (\cos\theta) + \sum_{n=1}^{\infty} \quad a_{2n} (2n+5-4\eta) \quad r^{2n} +$$

$$+ b_{2n} \quad r^{2n-2} \Big\} \cot\theta \frac{d}{d\theta} P_{2n} (\cos\theta) \tag{2.5}$$

$$\frac{\sigma r\theta (r,\theta)}{2\mu} = \int_0^{\infty} \frac{e}{r}^{-\xi r\cos\theta} \quad \cos\theta \Big[J_0 (\xi r \sin\theta)$$

$$\xi^2 r^2 \sin 2\theta \, A(\xi) + J_1 (\xi r \sin\theta) \Big\{ (2-2\eta) B(\xi) -$$

$$-A(\xi)(1+\xi r \cos\theta + \xi^2 r^2 \cos 2\theta) \Big\} \Big] d\xi +$$

$$+\sum_{n=1}^{\infty} \Big[\Big\{ a_{2n} (4n^2+4n+2\eta-1) + \frac{\alpha_t (1+\eta)}{2n-4\eta+5} \quad d_{2n} \Big\} r^{2n}$$

$$+ b_{2n} (2n-1) \quad r^{2n-2} \quad \Big] \frac{d}{d\theta} P_{2n} (\cos\theta) \tag{2.6}$$

where

$$A(\xi) = B(\xi) + \frac{m}{2} \frac{\varphi(\xi)}{\xi}$$

μ is modulus of rigidity, η Poisson's ratio, α_t the co-efficient of linear expansion,

$$m = \frac{1+\eta}{1-\eta} \alpha_t$$

3. THE TEMPERATURE FIELD

$$\text{Since } \frac{d}{d\theta} P_{2n} (\cos \theta) = 0, \text{ for } \theta = \pi/2$$

the conditions (1.4), (1.5) and $T(c, \theta) = 0$ are fulfilled if we can determine $\varphi(\xi)$ and d_{2n} such that the following equations are satisfied

$$\int_0^\infty \varphi(\xi) e^{-c\xi \cos\theta} J_0(c\xi \sin\theta) d\xi + \sum_{n=1}^\infty d_{2n} c^{2n} P_{2n}(\cos\theta) = 0 \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2} \qquad (3.1)$$

$$\int_0^\infty \xi \varphi(\xi) J_0(\xi r) d\xi = f(r) \qquad 0 \leqslant r \leqslant 1 \qquad (3.2)$$

$$= 0 \qquad 1 \leqslant r \leqslant c \qquad (3.3)$$

Hence

$$\varphi(\xi) = \int_0^1 r f(r) J_0(\xi r) dr \qquad (3.4)$$

Substituting this value of $\varphi(\xi)$ in (3.1) after changing the order of integration we get

$$\int_0^1 t f(t) \left[\int_0^\infty e^{-c\xi\cos\theta} J_0(\xi c \sin\theta) J_0(\xi t) d\xi \right] dt + \sum_{n=0}^\infty d_{2n} c^{2n} P_{2n}(\cos\theta) = 0$$

Substituting the value of the inner integral from appendix, we have

$$\sum_{n=0}^{\infty} \frac{(-)^n (2n)!}{2^{2n}[(n)!]^2} \int_0^1 \left[\left(\frac{t}{c}\right)^{2n+1} f(t)\, dt + \right.$$

$$\left. + d_{2n} c^{2n} \right] P_{2n} (\cos\theta) = 0 , \qquad 0 \leqslant \theta \leqslant \pi/2 \tag{3.5}$$

This implies that

$$d_{2n} c^{2n} + \frac{(-)^n \Gamma(n+\frac{1}{2})}{\Gamma \frac{1}{2}\ \Gamma(n+1)} \int_0^1 \left(\frac{t}{c}\right)^{2n+1} f(t)\, dt = 0 \tag{3.6}$$

The temperature in the plane of crack is given by

$$T(r, \pi/_2) = \int_0^{\infty} \varphi(\xi) J_0(\xi r)\, d\xi + \sum_{n=0}^{\infty} d_{2n} r^{2n} P_{2n}(0) =$$

$$= \int_0^1 t f(t) \left[\int_0^{\infty} J_0(\xi t) J_0(\xi r)\, d\xi \right] dt -$$

$$- \sum_{n=0}^{\infty} \int_0^1 \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{(1)_n\ n!} \left(\frac{rt}{c^2}\right)^{2n} \frac{t f(t)}{c}\, dt$$

$$T(r, \pi/_2) = \begin{cases} \displaystyle\int_0^r \frac{t f(t)}{r}\ {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{t^2}{r^2}\right) dt + \\ \displaystyle + \int_r^1 f(t)\ {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{r^2}{t^2}\right) dt - \\ \displaystyle - \int_0^1 {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{r^2 t^2}{c^4}\right) \frac{t f(t)}{c} & 0 \leqslant r < 1 \\ \displaystyle \int_0^1 \frac{t f(t)}{r c} \left[c\ {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{t^2}{r^2}\right) - r\ {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{r^2 t^2}{c^6}\right) \right] dt \\ & 1 \leqslant r \leqslant c \end{cases} \tag{3.7}$$

4. THE THERMOELASTIC FIELD

We divide the solution into two parts

Conditions of the crack face

Since $\frac{d}{d\theta} P_{2n} (\cos\theta) = 0$, for $\theta = \frac{\pi}{2}$

the condition (1.1) is satisfied, while the conditions (1.2) and (1.3) lead to dual integral equations

$$\int_0^\infty \xi B(\xi) J_0(\xi r) d\xi + \sum_{n=0}^{\infty} \Big\{ a_{2n} (4n^2 + 8n + 2\eta + 2)$$

$$(2n+1) r^{2n} + 4n^2 b_{2n} r^{2n-2} \Big\} P_{2n}(0)$$

$$= -\frac{m}{2} \Big[\int_0^\infty \varphi(\xi) J_0(\xi r) d\xi + \sum_{n=0}^{\infty} \frac{(2n+3)(2-2\eta)}{2n-4\eta+5}$$

$$d_{2n} r^{2n} P_{2n}(0) , \qquad 0 \leqslant r < 1 \qquad (4.1)$$

$$\int_0^\infty B(\xi) J_0(\xi r) d\xi = 0 \qquad 1 < r \leqslant c \qquad (4.2)$$

The second of these equations is satisfied if $B(\xi)$ is written in terms of an unknown function g(x) through the equation

$$B(\xi) = \int_0^1 g(x) \sin \xi x \, dx$$

$$= -g(1) \frac{\cos\xi}{\xi} + \frac{1}{\xi} \int_0^1 g'(x) \cos \xi x \, dx \qquad (4.3)$$

$$g(0) = 0$$

If we substitute this form in the first equation we get an Abel type integral equation, which on inverting gives

$$g(t) + \frac{2}{\pi} \sum_{n=o}^{\infty} (-)^n \left\{ a_{2n} (4n^2 + 8n + 2\eta + 2) \, t^{2n+1} + \right.$$

$$\left. + \, b_{2n} \, 2n \, t^{2n-1} \right\} = - \frac{m}{\pi} \left[\int_o^{\infty} \frac{\varphi(\xi)}{\xi} \sin \xi t \, d\xi + \right.$$

$$\left. + \sum_{n=o}^{\infty} \frac{(2-2\eta)(2n+3)(-)^n}{(2n+1)(2n+5-4\eta)} d_{2n} \, t^{2n+1} \right]$$

$$= - \frac{m}{\pi} \left[\int_o^1 x f(x) \left\{ \int_o^{\infty} \frac{\sin \xi t \; J_o(\xi x)}{\xi} d\xi \right\} dx - \right.$$

$$- \frac{t}{c} \sum_{n=o}^{\infty} \frac{(2-2\eta)(2n+3)}{(2n+1)(2n+5-4\eta)} \frac{\Gamma(n+\frac{1}{2})}{\Gamma \frac{1}{2} (n)!}$$

$$\left. \int_o^1 \left(\frac{xt}{c^2} \right)^{2n} x f(x) dx \right] \qquad (4.4)$$

In the above equation we have substituted the values of $\varphi(\xi)$ and d_{2n} from (3.4) and (3.6).

The conditions on the surface of sphere

The equation (4.4) gives one relation connecting the unknown function g(t) the co-efficients and the known funciion f(x). The remaining two relations are given by the conditions on the curved surface of the sphere. We shall deal with the two problems separately.

Problem 1

Here the conditions to be satisfied are

$$U_r (c, \theta) = \sigma_{r\theta} (c, \theta) = 0 \text{ for } 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

By substituting the values of $\varphi\ (\xi)$ and $B(\xi)$ from (3.4) and (4.3) in (2.2) and (2.6) after changing the order of integration, we obtain :

$$\int_0^1 g(x)\Big[\int_0^\infty e^{-c\xi\cos\theta}\{J_1(c\xi\sin\theta)\sin\theta\,(c\xi\cos\theta+2\eta-1)$$

$$+J_0(c\xi\sin\theta)(\cos\theta)(c\xi\cos\theta+2-2\eta)\}\ \sin\xi x\,d\xi\Big]dx+\frac{m}{2}$$

$$\int_0^1 x\,f(x)\Big[\int_0^\infty\{J_1(\xi c\sin\theta)\sin\theta\,(c\xi\cos\theta+1)+\cos^2\theta\,\xi c$$

$$J_0(\xi r\sin\theta)\}J_0(\xi x)\frac{d\xi}{\xi}\Big]dx+\sum_{n=0}^{\infty}\Big[\Big\{(a_{2n})(2n+1)(2n-2+4\eta)$$

$$+\frac{2\,a_t\,(1+\eta)}{2n+5-4\eta}\,d_{2n}\Big\}c^{2n+1}+2n\,b_{2n}\,c^{2n-1}\Big]P_{2n}(\cos\theta)=0$$

$$\int_0^1 g(x)\Big[\frac{d}{d\theta}\sin\theta\int_0^\infty\frac{e^{-\xi c\cos\theta}}{c}\cos\theta\,\{J_0(c\xi\sin\theta)\,c^2\xi^2\sin 2\theta$$

$$+J_1(c\xi\sin\theta)(1-2\eta-c\xi\cos\theta-c^2\xi^2\cos 2\theta)\}\sin\xi x\,d\xi\Big]dx$$

$$+\frac{m}{2}\int_0^1 x\,f(x)\Big[\frac{d}{d\theta}\sin\theta\int_0^\infty\frac{e^{-\xi c\cos\theta}}{c}$$

$$\cos\theta\,\{J_0(c\xi\sin\theta)\,\xi^2c^2\sin 2\theta-J_1(c\xi\sin\theta)\,(1+c\xi\cos\theta+c^2\xi^2$$

$$\cos 2\theta)\}\frac{J_0(\xi x)}{\xi}\,d\xi\Big]dx+\sum_{n=1}^{\infty}\Big[\Big\{a_{2n}(4n^2+4n+2\eta-1)+$$

$$\frac{a_t\,(1+\eta)}{2n-4\eta+5}\,d_{2n}\Big\}c^{2n}+(2n-1)\,b_{2n}\,c^{2n-2}\frac{d}{d\theta}\Big\{\sin\theta\frac{d}{d\theta}P_{2n}(\cos\theta)\Big\}\Big]=0$$

Substituting the values of the inner integrals from appendix and using the fact the Legendre polynomials are orthogonal, we get the following two equations connecting the unknown coefficients a_{2n} and b_{2n} with g (x) and the known function f (x). These equations are

$$\left\{a_{2n}\,(2n+1)\,(2n+4\eta-2)+\frac{2\,\alpha_t\,(1+\eta)}{2\,n-4\eta+5}\,d_{2n}\right\}c^{2n+1}+2n\,b_{2n}\,c^{2n-1}$$

$$+\frac{(-)^n}{c}\int_0^1 g(x)\left\{\left(\frac{x}{c}\right)^{2n+1}\frac{2n+2\eta+(2n+1)\,(2n+2)}{4n+3}-\right.$$

$$\left.\left(\frac{x}{c}\right)^{2n-1}\frac{2n\,(2n-4\eta+3)}{4n-1}\right\}+\frac{m}{2}\,\frac{(-)^n\,\Gamma\,(n+\frac{1}{2})}{\Gamma\,\frac{1}{2}\,\Gamma\,(n+1)}$$

$$+\int_0^1 x\,f\,(x)\left\{\frac{2n-1}{4n-1}\left(\frac{x}{c}\right)^{2n}-\frac{2n+1}{4n+3}\left(\frac{x}{c}\right)^{2n+2}\right\}dx=0$$

$$\left\{a_{2n}\,(4n^2+4n+2\eta-1)\,2n\,(2n+1)+\right.$$

$$\left.\frac{\alpha_t\,(1+\eta)\,2n\,(2n+1)}{2n+5-2\eta}\,d_{2n}\right\}c^{2n+1}+b_{2n}\,2n\,(4n^2-1)\,c^{2n-1}-$$

$$\frac{(-)^n}{c}\int_0^1 g\,(x)\left[\frac{2n\,(2n+2)}{4n+3}\,\{1-2\eta-(2n+1)\,(2n+3)\}\right.$$

$$\left(\frac{x}{c}\right)^{2n+1}+\frac{2n\,(2n+1)}{4n-1}\,\{4n^2+2\eta-2\}\left(\frac{x}{c}\right)^{2n+1}dx-$$

$$\frac{m}{2}\,\frac{(-)^n\,\Gamma\,(n+\frac{1}{2})}{\Gamma\,\frac{1}{2}\,\Gamma\,(n+1)}\int_0^1 x\,f\,(x)\left[\frac{2n\,(2n+1)\,(2n+2)}{4n+3}\left(\frac{x}{c}\right)^{2n+2}-\right.$$

$$\left.\frac{4n^2\,(2n+1)}{4n-1}\left(\frac{x}{c}\right)^{2n}\right]dx=0$$

With a view to simplify the calculations we assume that $\eta = \frac{1}{4}$. From these eqations we deduce that

$$(2n+1)\,a_{2n}\,c^{2n}=-\frac{(-)^n}{c^2}\left[\int_0^1\frac{g\,(x)}{(4n-1)\,(3n+1)}\left\{\frac{4n+1}{4n+3}\left(4n^2+\right.\right.\right.$$

$$\left.\left.\left.8\,n+\frac{5}{2}\right)\left(\frac{x}{c}\right)^{2n+1}+n\,(2n+1)\left(\frac{x}{c}\right)^{2n-1}\right\}d\,x\right]+$$

$$\frac{(-)^n m \Gamma(n+\frac{1}{2})}{2c\,\Gamma(\frac{1}{2})\,(n)!\,(3n+1)\,(4n-1)} \int_0^1 x f(x)$$

$$\left[\frac{4n+1}{4n+3}\left(\frac{x}{c}\right)^{2n+2} - \left(\frac{x}{c}\right)^{2n}\right] dx +$$

$$\frac{2\,(1+\eta)\,a_t\,(n-1)\,d_{2n}}{(2n+4)\,(4n-1)\,(3n+1)}$$

$$2n\,b_{2n}\,c^{2n-2} = -\frac{(-)^n}{c^2}\left[\int_0^1 \frac{g(x)}{(4n-1)(3n+1)}\left\{n\left(4n^2+8n+\frac{5}{2}\right)\right.\right.$$

$$\left.\left.\left(\frac{x}{c}\right)^{2n+1} - n\,(4n+1)\,(4n+3)\left(\frac{x}{c}\right)^{2n-1}\right\} dx\right] +$$

$$\frac{m\,\Gamma(n+\frac{1}{2})}{2c\,\Gamma(\frac{1}{2})\,(n)!\,(4n+1)\,(3n+1)}\left[\int_0^1 x f(x)\left\{n\,(2n+1)\right.\right.$$

$$\left.\left.\left(\frac{x}{c}\right)^{2n+2} - n\,(2n-1)\left(\frac{x}{c}\right)^{2n}\right\} dx\right] - \frac{4\,a_t\,(1+\eta)\,n^2}{(4n-1)\,(3n+1)} d_{2n}\,c^{2n}$$

Problem 2

In this case the curved surface of the sphere is free from traction. The conditions to be satisfied are,

$$\sigma_{rr}\,(c,\theta) = \sigma_{r\theta}\,(c,\theta) = 0 \text{ for } 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

A procedure similar to that followed in problem one leads to the relations

$$(8n^2+6n+5/2)\,a_{2n}\,c^{2n} = -(-)^n\,\frac{(2n+2)}{c^2}\int_0^1 g(x)$$

$$\left\{\frac{(4n+1)\,(4n^2+8n+5/2)}{(2n+1)\,(4n+3)}\left(\frac{x}{c}\right)^{2n+1} - 2n\left(\frac{x}{c}\right)^{2n-1}\right\} dx +$$

$$+\frac{(-)^n m}{2c}\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma\frac{1}{2}\,(n)!}\,(2n+2)\int_0^1 x\,f(x)\left[\frac{4n+1}{4n+3}\left(\frac{x}{c}\right)^{2n+2} -\right.$$

$$- \left(\frac{x}{c}\right)^{2n} \Big] dx - \frac{3\,\alpha_t\,(1+\eta)}{2n+4}\, d_{2n}\, c^{2n}$$

$$2n\, b_{2n}\, c^{2n-2} = \frac{(-)^n}{c^2} \int_0^1 g(x) \left[\frac{2n\,(2n+2)\,(4n^2+8n+5/2)}{8n^2+6n+5/2} \left(\frac{x}{c}\right)^{2n+1} \right.$$

$$- \left(\frac{x}{c}\right)^{2n-1} \left\{ \frac{2n\,(4n^2+6n-\frac{1}{2})}{(2n-1)\,(4n-1)} + \right.$$

$$\left. \left. + \frac{2n\,(2n+2)\,(4n^2-2n-5/2)\,(2n+1)}{(2n+1)\,(8n^2+6n+5/2)} \right\} \right] dx -$$

$$- \frac{(-)^n\, m}{2c} \frac{\Gamma\,(n+\frac{1}{2})}{\Gamma\,\frac{1}{2}\,\Gamma\,(n+1)} \int_0^1 \frac{x\, f(x)}{2n-1} \left[\frac{(2n+1)\,(2n+2)}{4n+3} \left\{ 1 + \right. \right.$$

$$\left. + \frac{(4n+1)\,(4n^2-2n-5/2)}{8n^2+6n+5/2} \right\} \left(\frac{x}{c}\right)^{2n+2} -$$

$$\left. - \left(\frac{x}{c}\right)^{2n} \left\{ \frac{4n^2+6n-2}{4n-1} + \frac{(2n+1)\,(2n+2)\,(4n^2-2n-5/2)}{8n^2+6n+5/2} \right\} \right] dx$$

$$+ \frac{2\alpha_t\,(1+\eta)\, n\, d_{2n}}{8n^2+6n+5/2}\, c^{2n}\,.$$

By substituting the values of co-efficients a_{2n} and b_{2n} and that of d_{2n} from (3.6) in equation (4.4) we obtain a Fredholm integral equation of the second kind

$$g(t) + \frac{2}{\pi c} \int_0^1 g(x)\, K(xt/c^2)\, dx = - \frac{m}{2} \left[\int_0^1 \frac{\varphi\,(\xi)\,\sin \xi\, t}{\xi}\, d\xi + \right.$$

$$\left. + \frac{t}{c} \int_0^1 x\, f(x)\, N(xt/c^2)\, dx \right]$$

where for the first problem

$$K\left(\frac{xt}{c^2}\right) = \sum_{n=o}^{\infty} \left[\frac{2n}{3} + \frac{11}{18} + \frac{1}{6(2n+1)} - \frac{5}{252(3n+1)} - \frac{3}{28(4n+3)} - \right.$$

$$- \frac{361}{84(3n+4)} - \frac{5}{63(3n+4)} - \frac{x^2+t^2}{c^2} \left\{ \frac{n}{3} + \frac{35}{36} + \right.$$

$$\left. \left. + \frac{5}{126(3n+4)} + \frac{19}{28(4n+3)} \right\} \right] \left(\frac{xt}{c^2}\right)^{2n+1}$$

$$N\left(\frac{xt}{c^2}\right) = \sum_{n=o}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma\frac{1}{2}\,\Gamma(n+1)} \left[- \left\{ \frac{3(2n+3)}{2(2n+1)(2n+4)} \right. \right.$$

$$\left. + \frac{(4n^2+8n+5/2)(5n+1)}{(2n+4)(3n+1)(4n-1)} \right\} + \left(\frac{x}{c}\right)^2 \left\{ \frac{(2n+1)(4n+5)}{2(3n+4)(4n+3)} \right.$$

$$\left. \left. + \frac{(4n^2+8n+5/2)(4n+1)}{(3n+1)(4n-1)(4n+3)} \right\} + \left(\frac{x}{c}\right)^4 \frac{(2n+1)(2n+3)}{2(3n+4)(4n+3)} \right] \left(\frac{xt}{c^2}\right)^{2n}$$

and for the second problem

$$K\left(\frac{xt}{c^2}\right) = \frac{x^2+t^2}{c^2} \sum_{n=1}^{\infty} \left\{ 2n^2 + \frac{9n}{2} - \frac{356}{16(2n+1)} - \right.$$

$$\left. - \frac{60n+75}{16(16n^2+12n+5)(2n+1)} + \frac{5}{4} \right\} \left(\frac{xt}{c^2}\right)^{2n-1} -$$

$$- \sum_{n=0}^{\infty} \left\{ 4n^2 + 13n + \frac{19}{2} + \frac{4}{3(2n+1)} + \frac{5(29n+20)}{6(16n^2+12n+5)} - \right.$$

$$\left. - \frac{15(n+1)}{2(16n^2+44n+33)} \right\} \left(\frac{xt}{c^2}\right)^{2n+1}$$

$$N\left(\frac{xt}{c^2}\right) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+1)} \left(\frac{xt}{c^2}\right)^{2n}$$

$$\left[\left\{ \frac{9\,(4n^2 + 8n + 5/2)}{(2n+4)\,(16n^2 + 12n + 5)} - \frac{(2n+2)\,(8n^2+16n+5)}{16n^2+12n+5} - \frac{6n+9}{(4n+2)\,(2n+4)} \right\} + \right.$$

$$\left(\frac{x}{c}\right)^2 \frac{(2n+4)\,(4n+1)\,(8n^2+16n+5)}{(4n+3)\,(16n^2+12n+5)} + \frac{4n^2+14n+8}{(2n+2)\,(4n+3)} +$$

$$\frac{(2n+3)\,(2n+4)\,(8n^2+12n-1)}{(2n+2)\,(16n^2+44n+33)} - \frac{6n+3}{16n^2+44n+33} -$$

$$\left. \left(\frac{x}{c}\right)^4 \frac{(2n+3)\,(2n+4)}{(2n+2)\,(4n+7)} \left\{ 1 + \frac{(4n^2+5)\,(8n^2+12n-1)}{16n^2+44n+33} \right\} \right]$$

If c is sufficiently large the above integral equation has a unique solution. Though the precise range of values of c, for which the above equation has a solution, is hard to determine, it is believed that for all values of 1/c less than unity, the integral equation has a unique solution. Various series occuring in the expressions for the functions K and N are convergent for $xt/c^2 < 1$. Hence the kernel of (4.5) is bounded and L^2 (0, 1). Consequently, for small values of 1/c, an approximate analytical solution can be obtained by replacing the kernel by a polynomial in x and t. In general, recourse has to be made to the numerical solution.

Iterative Solution :

When c is sufficiently large, we can write

$$K\left(\frac{xt}{c^2}\right) = A_0 \frac{xt}{c^2} + A_1 \frac{xt\,(x^2+t^2)}{c^4} + A_2 \frac{x^3 t^3}{c^6} + 0\,(c^{-8}) \qquad (4.6)$$

$$N\left(\frac{xt}{c^2}\right) = B_0 + B_1 \frac{x^2}{c^2} + \frac{x^2}{c^4}\,(B_2 x^2 + B_3 t^2) + B_4 \frac{x^4 t^2}{c^6} + 0\,(c^{-8}) \qquad (4.7)$$

where the values of the constants for the two problems are

	Problem 1	Problem 2		Problem 1	Problem 2
A_0	5.0000	— 7.2727	B_1	— 0.6250	2.0909
A_1	— 1.2083	7.0303	B_2	0.1250	— 0.7273
A_2	— 0.1308	— 25.6911	B_3	— 0.8125	— 1.8742
B_0	— 0.5000	— 2.0000	B_4	0.5693	3.0651

An iterative solution is obtained by writing

$$g(t) = \frac{m}{2}\left[g_0(t) + \frac{g_1(t)}{c} + \dots\dots\dots + \frac{g_7(t)}{c^7} + 0(c^{-8}) \right]$$

where the functions $g_n(t)$ are obtained in terms of the known function f(x)

$$g_0(t) = -\int_0^1 \frac{\varphi(\xi)}{\xi} \sin \xi t \, d\xi \ ,$$

$$g_1(t) = -B_0 t \int_0^1 x f(x) \, dx \ , \qquad g_2(t) = 0 \ ,$$

$$g_3(t) + \frac{2A_0 t}{\pi} \int_0^1 x \, g_0(x) \, dx = -B_1 t \int_0^1 x^3 f(x) \, dx$$

$$g_4(t) + \frac{2A_0 t}{\pi} \int_0^1 x \, g_1(x) \, dx = 0$$

$$g_5(t) + \frac{2A_1 t}{\pi} \int_0^1 x(x^2 + t^2) \, g_0(x) \, dx = -t \int_0^1 x^3(B_2 x^2 + B_3 t^2) f(x) \, dx$$

$$g_6(t) + \frac{2t}{\pi} \int_0^1 \left\{ A_1(x^2 + t^2) \, g_1(x) + A_0 \, g_3(x) \right\} x \, dx = 0$$

$$g_7(t) + \frac{2t}{\pi} \int_0^1 \left\{ A_0 \, g_4(x) + A_2 x^2 t^2 \, g_0(x) \right\} x \, dx = - B_4 t^3 \int_0^1 x^5 f(x) \, dx$$

$$g_8(t) + \frac{2t}{\pi} \int_0^1 \left\{ A_0 \, g_5(x) + (x^2 + t^2) A_1 \, g_3(x) + x^2 t^2 \, g_1(x) \right\} x \, dx = 0$$

5. CRACK OPENED BY CONSTANT FLUX

Quantities of Physical Interest

We now consider the physically important case when the prescribed flux is constant. Then we have

$$f(r) = -\mathcal{J}_0$$

hence

$$\varphi(\xi) = -\mathcal{J}_0 \frac{J_1(\xi)}{\xi}, \quad g_0(t) = \frac{\mathcal{J}_0}{2}\left[\sin^{-1} t + t\sqrt{1-t^2}\right]$$

The temperature in the plane $\theta = \pi/2$ is given by

$$T(r, \pi/2) = \begin{cases} \frac{\mathcal{J}_0}{2c}\left\{ {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 2; \frac{r^2}{c^4}\right) - 2c\, {}_2F_1\left(\tfrac{1}{2}, -\tfrac{1}{2}; 1; r^2\right)\right\} & 0 \leq r < 1 \\ \frac{\mathcal{J}_0}{2cr}\left\{\gamma\, {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 2; \frac{r^2}{c^4}\right) - c\, {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 2; \frac{1}{r^2}\right)\right\} & 1 < r \leq c \end{cases} \quad (5.1)$$

Hypergeometric functions in the above expression can be expressed in terms of elliptic integrals of the first and the second kind.

The iterative solution in section 4 can be written as

Problem 1

$$g(t) = \frac{m\mathcal{J}_0}{2}\Big\{ \tfrac{1}{2}(\sin^{-1} t + t\sqrt{1-t^2}) - (.25x + 1.0137\, x^3 -$$
$$- .3857\, x^4 - .6260\, x^5 - .7719\, x^6 + .2635\, x^7 - 1.1010\, x^8)t -$$
$$- (1.2986\, x^5 + .0680\, x^6 - .0949\, x^7 + .2793\, x^8)\, t^3 + 0(x^9)\Big\} \quad (5.2)$$

Problem 2

$$g(t) = \frac{m\mathcal{J}_0}{2}\Big\{ \tfrac{1}{2}(\sin^{-1} t + t\sqrt{1-t^2}) - (x - 1.9621\, x^3 -$$
$$- 1.5666\, x^4 - .6742\, x^5 - 4.1054\, x^6 - 1.8044\, x^7 -$$
$$- .1723\, x^8)t + .9546\, x^5 + 1.5223\, x^6 + .5107\, x^7 -$$
$$- .3634\, x^8)\, t^3 + 0(x^9)\Big\} \quad (5.3)$$

where $x = 1/c$.

With the help of these solutions expressions for the normal components of displacement and stress intensity factor can be obtained.

The normal component of displacement for $0 \leq r < 1$ is given by

$$U_\theta (r, \pi/2) = -(2-2\eta) \int_0^\infty B(\xi)\, J_0(\xi r)\, d\xi =$$

$$= -(2-2\eta) \int_r^1 \frac{g(t)}{\sqrt{t^2 - r^2}}\, dt \tag{5.4}$$

The stress intens.ty actor is defined by the equation

$$N = \lim_{r \to 1} (r-1)^{\frac{1}{2}} \left[\sigma_{\theta\theta} (r, \pi/2) \right]_{1 < r \leq c}$$

It is easy to show that

$$\sigma_{\theta\theta} (r, \pi/2) = 2\mu \left[\frac{g(1)}{\sqrt{(r^2 - 1)}} + 0(1) \right] \tag{5.5}$$

Hence

$$N = \sqrt{2}\, \mu\, g(1) \tag{5.6}$$

The stress intensity factor for the first problem is

Problem 1

$$N = \frac{m \mathcal{J}_0 \mu}{\sqrt{2}} \left[\frac{\pi}{4} - .25x - 1.0137\, x^3 + .3651\, x^4 - .6726\, x^5 + \right.$$

$$\left. + .7039\, x^6 - .1686\, x^7 + .8217\, x^8 + 0(x^9) \right] \tag{5.7}$$

Problem 2

$$N = \frac{m \mathcal{J}_0 \mu}{\sqrt{2}} \left[\frac{\pi}{2} - x + 1.9621\, x^3 + 1.5666\, x^4 + 1.6288\, x^5 + \right.$$

$$\left. + 5.6277\, x^6 + 2.3151\, x^7 - .1911\, x^8 + 0(x^9) \right] \tag{5.8}$$

6. Numerical Solution :

The iterative solution g(t) of the integral equation (4.5) and the formulae for the stress intensity factor presented in the last section are of value only if the radius of the sphere is sufficiently large. When $x = 1/c$ is slightly less than unity, the equation has to be solved numerically. We assume that

$$g(x) = \frac{m \mathfrak{J}_0}{2} g^* (x)$$

Then (4.5) reduces to the form

$$g^*(t) + 2 \int_0^1 g^*(x) K\left(\frac{xt}{c^2}\right) dx = \tfrac{1}{2} \left\{ \sin^{-1} t + \right.$$

$$\left. + t\sqrt{1 - t^2} \right\} + \frac{t}{c} \int_0^1 x N\left(\frac{xt}{c^2}\right) dx \qquad (6.1)$$

The integral equation was solved for $1/c = .6, .7, .8$ and $.9$. In order to apply the method of Fox and Goodwin [2] we replace the integral equation (6.1) by a set of ten simultaneous equations. The solution of these equations is nothing but the solution of the integral equation under consideration. The kernel $K(xt/c^2)$ is expressed in terms of elementary functions [3] whose values can be easily calculated. It is easy to calculate the right hand side numerically. The results of these calculations are shown in Table 1 and Figure 1 and 2.

The stress intensity factory is

$$N = \frac{m \mu \mathfrak{J}_0}{\sqrt{2}} \quad g^*(1)$$

Substituting the value of $g^*(1)$ from table for $x = 1/c = 0.6$, we get

$$N = \frac{m \mu \mathfrak{J}_0}{\sqrt{2}} \quad 0.476925$$

for the first problem and

$$N = \frac{m \mu \mathfrak{J}_0}{\sqrt{2}} \quad 1.479294.$$

The normal component of displacement was calculated from the equation (5.4). The results of these computations are shown in figure 3.

TABLE I

x/t t ↓	x → Problem 1				Problem 2			
	0.6	0.7	0.8	0.9	0.6	0.7	0.8	0.9
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.069597	0.058734	0.046054	0.036602	0.174780	0.198137	0.243780	0.271680
0.2	0.135225	0.116362	0.96816	0.077544	0.340449	0.395383	0.472858	0.534070
0.3	0.195690	0.169449	0.143475	0.114501	0.502732	0.589271	0.694555	0.792331
0.4	0.261299	0.227082	0.194094	0.159745	0.667013	0.777582	0.912744	1.047522
0.5	0.320831	0.278337	0.236526	0.194791	0.828623	0.959201	1.127313	1.302835
0.6	0.375295	0.324485	0.273606	.222382	0.985055	1.131879	1.334841	1.553945
0.7	0.420513	0.361560	0.303974	0.241948	1.131832	1.291708	1.528424	1.795697
0.8	0.456263	0.388853	0.325573	0.250202	1.264375	1.435304	1.697501	2.019129
0.9	0.475764	0.397912	0.333037	0.251228	1.380217	1.553062	1.828530	2.206656
1.0	0.476925	0.383217	0.318415	0.247026	1.479294	1.634517	1.904951	2.337334

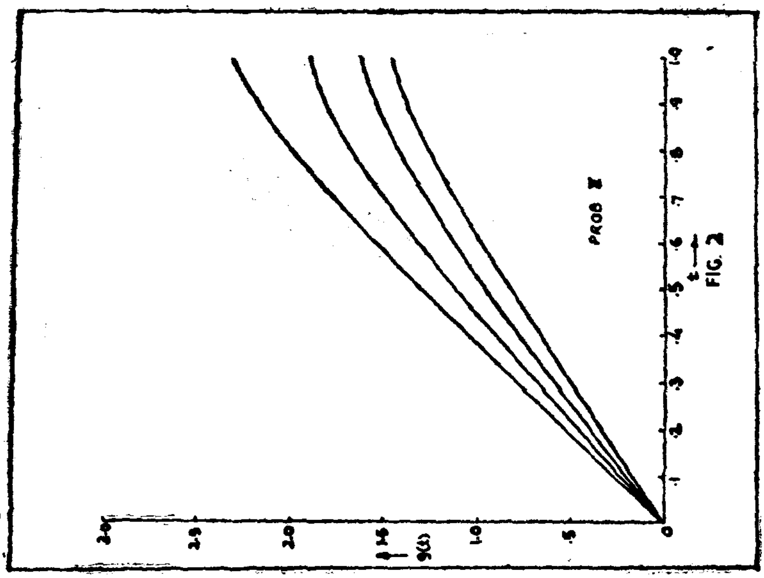
PROB II
y(t)
t
0
.5
1.0
1.5
2.0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0

FIG. 2

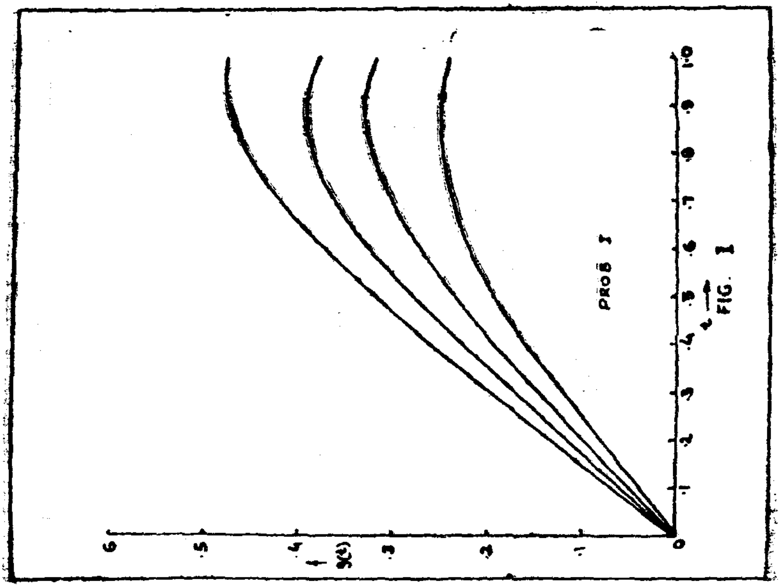
PROB I
y(t)
t
0
.1
.2
.3
.4
.5
.6
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0

FIG. 1

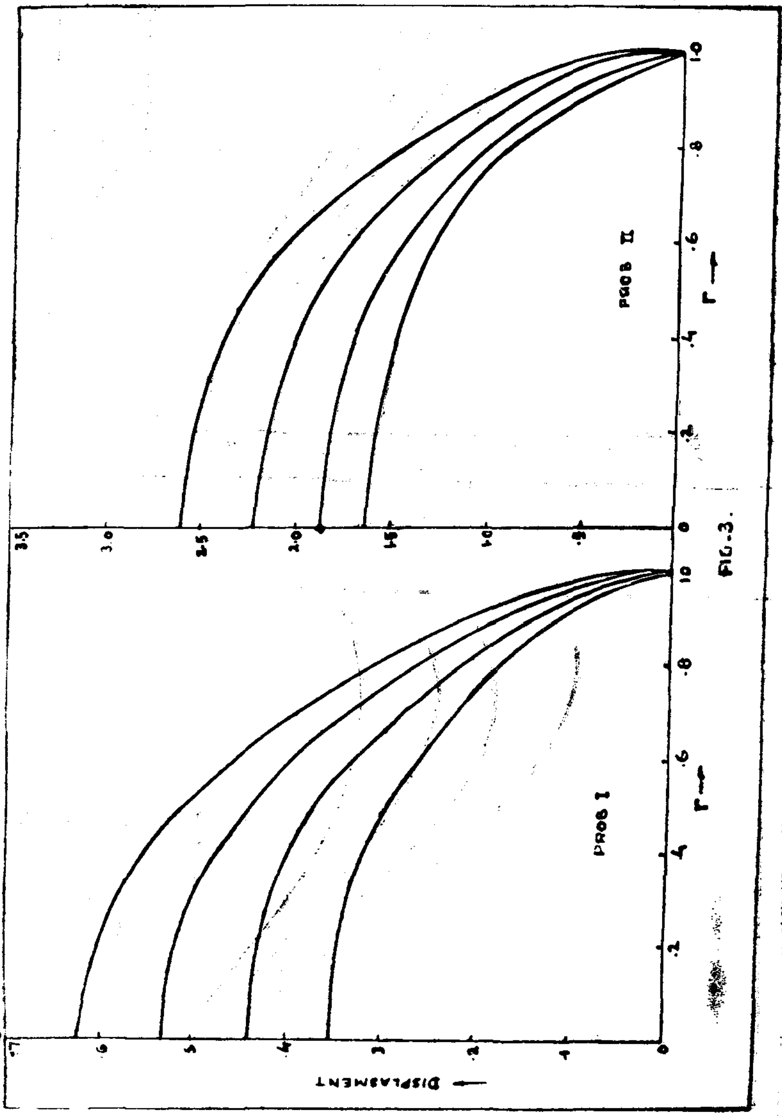
DISPLACEMENT
PROB I
PROB II
r
0
.2
.4
.6
.8
1.0
.1
.2
.3
.4
.5
.6
.7
.5
1.0
1.5
2.0
2.5
3.0
3.5

FIG. 3.

APPENDIX

The integrals given here are easily derived from the well known result [4]

$$\int_0 x^{m+1} e^{-ax} J_\nu (x y) \, dx = (-)^{m+1} y^{-\nu} \frac{d^{m+1}}{d a^{m+1}}$$

$$\left\{ (a^2 + y^2)^{-\frac{1}{2}} \left[a^2 + y^2) - a \right] \right\} \qquad \nu > -m-2$$

and the well known recurrence relations for Legendre polynomials. The method used for deriving these integrals is similar to that given in Sneddon's book on integral transforms [5]. We have for $x < c$

$$\int_0^\infty e^{-c \xi \cos \theta} J_0 (\xi c \sin \theta) J_0 (\xi x) \, d\xi =$$

$$= \sum_{n=0}^\infty \frac{(-)^n \Gamma(n + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(n+1)} \left(\frac{x}{c}\right)^{2n} \frac{P_{2n}(\cos \theta)}{c}$$

$$\int_0^\infty \frac{e^{-c \xi \cos \theta}}{\xi} \left[J_1 (\xi c \sin \theta) \sin \theta \, (1 + \xi \cos \theta) + \right.$$

$$\left. + \cos^2 \theta \, \xi \, c \, J_0 (\xi c \sin \theta) \right] J_0 (\xi x) \, d\xi =$$

$$= \sum_{n=0}^\infty \frac{(-)^n \Gamma(n + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(n+1)} \left[\frac{2n-1}{4n-1} \left(\frac{x}{c}\right)^{2n} - \right.$$

$$\left. - \frac{2n+1}{4n+3} \left(\frac{x}{c}\right)^{2n+2} \right] P_{2n}(\cos \theta)$$

$$\frac{d}{d\theta} \sin \theta \left[\int^\infty e^{-c \xi \cos \theta} \cos \theta \left\{ J_0 (\xi c \sin \theta) c^2 \xi^2 \sin 2\theta - \right. \right.$$

$$\left. \left. - J_1 (\xi c \sin \theta) (1 + c \xi \cos \theta + c^2 \xi^2 \cos 2\theta) \right\} J_0 (\xi x) \, d\xi \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-)^n \Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} \Big[\frac{2n(2n+1)(2n+2)}{4n+3} \left(\frac{x}{c}\right)^{2n+2}$$

$$- \frac{4n^2(2n+1)}{4n-1} \left(\frac{x}{c}\right)^{2n} \Big] P_{2n}(\cos\theta)$$

$$\int_0^{\infty} \frac{e^{-c\xi\cos\theta}}{\xi} \Big[J_1(c\xi\sin\theta) \Big\{ 1 + c\xi\cos\theta +$$

$$+ 2c^2\xi^2\cos^2\theta \Big\} \sin\theta + J_0(c\xi\sin\theta)\,\xi\, c\,(1 +$$

$$+ c\xi\cos\theta\cos 2\theta) \Big] J_0(\xi x)\, d\xi$$

$$= \sum_{n=0}^{\infty} \frac{(-)^n \Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} \Big[\frac{4n^2+6n-2}{4n-1} \left(\frac{x}{c}\right)^{2n} -$$

$$- \frac{(2n+1)(2n+2)}{4n+3} \left(\frac{x}{c}\right)^{2n+2} \Big] P_{2n}(\cos\theta).$$

The other results used in section 4 are given in [1].

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