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STEADY STATE HEAT FLOW IN A SHELL ENCLOSED BETWEEN TWO OBLATE SPHEROIDS

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1. INTRODUCTION

Cases in which heat is produced in the solids are important in technical applications [1 ; pp. 12-13]. Heat may be produced by the passage of an electric current, dielectric or from radiation, mechnical generation in viscous or plastic flow, chemical reaction, hyderation of cement and the ripening of apples. Nuclear reactors and space research also give rise to different problems of heat transfer.

Again temperature distribution within a spherical shell is known [3 ; pp. 329]. A similar problem for the oblate spheroidal shell does not appear to have been solved.* Here in this paper we have solved this problem.

Oblate spheroidal co-ordinates α, β, φ are related to the rectangular co-ordinates x, y, z by the relations [2 ; pp. 214]

$$x = c \cosh \alpha \sin \beta \cos \varphi, \quad y = c \cosh \alpha \sin \beta \sin \varphi, \quad z = c \sinh \alpha \cos \beta, \tag{1.1}$$

where

$$0 \leqslant \alpha < \infty, 0 \leqslant \beta \leqslant \pi, -\pi < \varphi \leqslant \pi,$$

and $c > 0$ is a scale factor. Every point of space is characterized by a unique triple of numbers α, β, φ. The corresponding triply orthogonal system of surfaces consists of the oblate spheroides α = const, the single-sheeted hyperboloids of revolution β = const and the planes φ = const.

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* The corresponding problem for prolate spheroidal shell has been solved by Bhonsle [4 ; pp. 101].

2. STATEMENT OF THE PROBLEM

We consider the temperature distribution for the set of points I $(\alpha_1 < \alpha < \alpha_2)$ when β_1 $(\alpha = \alpha_1)$ is kept at the temperature $u(\alpha, \beta) = f(\beta)$ and β_2 $(\alpha = \alpha_2)$ is kept at the temperature $u(\alpha_2, \beta) = 0$. Thus we have

(2.1) $$u(\alpha, \beta) \mid_{\beta_1(\alpha=\alpha_1)} = f(\beta),$$

(2.2) $$u(\alpha, \beta) \mid_{\beta_2(\alpha=\alpha_2)} = 0.$$

3. SOLUTION OF THE PROBLEM

Because of the azimuthal symmetry the equation satisfied by the temperature function $u(\alpha, \beta)$ will be

(3.1) $$\frac{1}{\cosh \beta} \frac{\partial}{\partial \alpha} \left(\cosh \alpha \frac{\partial u}{\partial \alpha} \right) + \frac{1}{\cos \beta} \frac{\partial}{\partial \beta} \left(\cos \beta \frac{\partial u}{\partial \beta} \right) = 0.$$

Its solution is [2 ; pp. 217]

(3.2) $$U(\alpha, \beta) = \sum_{n=0}^{\infty} \left[M_n P_n (i \sinh \alpha) + N_n Q_n (i \sinh \alpha) \right] P_n (\cos\beta).$$

Because of the boundary condition (2.2) we have

(3.3) $$N_n = - M_n \frac{P_n (i \sinh \alpha_2)}{Q_n (i \sinh \alpha_2)}$$

Substituting (3.3) in (3.2)

(3.3) $U(\alpha, \beta) =$

$$\sum_{n=1}^{\infty} M_n \frac{P_n (i \sinh \alpha) Q_n (i \sinh \alpha_2) - P_n (i \sinh \alpha_2) Q_n (i \sinh \alpha)}{Q_n (i \sinh \alpha_2)} . P_n (\cos \beta).$$

Because of the boundary condition (2.1)

(3.5) $$f(\beta) = \sum_{n=0}^{\infty} M_n P_n (\cos \beta) \times$$

$$\times \frac{P_n (i \sinh \alpha_1) Q_n (i \sinh \alpha_2) - P_n (i \sinh \alpha_2) Q_n (i \sinh \alpha_1)}{Q_n (i \sinh \alpha_2)} \qquad 0 < \beta < \pi.$$

Now making use of the orthogonal property of the Legendre polynomials [1;pp. 217] we shall get

$$(3.6)\ M_n = \frac{(n+\frac{1}{2})\ Q_n\ (i \sinh \alpha_2) \int_0^\pi f(\beta)\ P_n (\cos \beta) \sin \beta\, d\beta}{P_n (i \sinh \alpha_1)\ Q_n (i \sinh \alpha_2) - Q_n (i \sinh \alpha_1)\ P_n (i \sinh \alpha_2)}$$

Substituting (3.6) in (3.4), we get the solution as

(3.7) $U(\alpha, \beta) =$

$$= \sum_{n=0}^{\infty} \frac{A_n\ P_n (i \sinh \alpha)\ Q_n (i \sinh \alpha_2) - P_n\ (i \sinh \alpha_2)\ Q_n (i \sinh \alpha)}{P_n\ (i \sinh \alpha_1)\ Q_n (i \sinh \alpha_2) - P_n (i \sinh \alpha_2)\ Q_n (i \sinh \alpha)}$$

where

$$(3.8) \qquad A_n = (n+\tfrac{1}{2}) \int_0^\pi f(\beta)\ P_n (\cos \beta) \sin \beta\, d\beta$$

and provided it is convergent.

4. CONVERGENCE OF INFINITE SERIES.

Let us write (3.7) as

$$(4.1) \quad U(\alpha, \beta) = \sum_{n=0}^{\infty} U_n$$

and (3.8) as

$$(4.2) \qquad A_n = (n+\tfrac{1}{2})\, R_n.$$

Where let us assume that R_n is bounded and monotonically decreasing. We shall make use of the following results [1; pp 191]

$$(4.3)\ P_\nu\ (\cosh \alpha) = \frac{e^{(\nu+\frac{1}{2})\alpha}}{(2\nu\pi \sinh \alpha)^{\frac{1}{2}}}\ [1+O(|\nu|)^{-1}.]$$

$$|\nu| \to \infty,\ |\arg \nu| \leqslant \frac{\pi}{2} - \delta,\quad 0 < \alpha_0 \leqslant \alpha \leqslant \alpha_1 < \infty.$$

$$(4.4)\ Q_n\ (\cosh \alpha) = \frac{\pi^{\frac{1}{2}}}{(2\nu\pi \sinh \alpha)^{\frac{1}{2}}}\ e^{-(\nu+\frac{1}{2})\alpha}\ [1+O(|\nu|)^{-1}].$$

$$|\nu| \to \infty, \ |\arg \nu| \leqslant \frac{\pi}{2} - \delta, \ 0 < \alpha_0 \leqslant \alpha < \infty.$$

(4.5) $P_n(\cos \beta) \leqslant 1.$

Therefore from (3.7) and (4.1) we have

$$U = \frac{(n+\frac{1}{2}) R_n \, [P_n(i \sinh \alpha) \, Q_n(i \sinh \alpha_1) - P_n(i \sinh \alpha_1) \, Q_n(i \sinh \alpha)}{P_n(i \sinh \alpha_2) \, Q_n(i \sinh \alpha_1) - P_n(i \sinh \alpha_1) \, Q_n(i \sinh \alpha_2)}$$

$$= R_n (n+\tfrac{1}{2}) \times$$

$$\times \frac{\dfrac{e^{(n+\frac{1}{2})(\alpha-\alpha_1)} + O\left(\frac{1}{n}\right)}{2\, ni \sqrt{\cosh \alpha \cosh \alpha_1}} - \dfrac{e^{-(n+\frac{1}{2})(\alpha-\alpha_1)} + O\left(\frac{1}{n}\right)}{2\, ni \sqrt{\cosh \alpha \cosh \alpha_1}}}{\dfrac{e^{(n+\frac{1}{2})(\alpha_2-\alpha_1)} + O\left(\frac{1}{n}\right)}{2\, ni \sqrt{\cosh \alpha_1 \cosh \alpha_2}} - \dfrac{e^{-(n+\frac{1}{2})(\alpha_2-\alpha_1)} + O\left(\frac{1}{n}\right)}{2\, ni \sqrt{\cosh \alpha_1 \cosh \alpha_2}}},$$

$$= \frac{(n+\frac{1}{2}) R_n \sqrt{\cosh \alpha_2}}{\sqrt{\cosh \alpha}} \; \frac{[e^{(n+1/2)(-\alpha_1+\alpha)} - e^{-(n+1/2)(\alpha_1+\alpha)} + O(1/n)]}{[e^{(n+1/2)(-\alpha_1+\alpha_2)} - e^{-(n+1/2)(-\alpha_1+\alpha_2)} + O(1/n)]}$$

$$= \frac{(n+\frac{1}{2}) R_n \sqrt{\cosh \alpha_2}}{\sqrt{\cosh \alpha}} \; e^{-(n+1/2)(\alpha_2-\alpha)}$$

$$\frac{1 - e^{-(2n+1)(\alpha-\alpha_1)} + 0\{1/n \; e^{-(n+1/2)(\alpha-\alpha_1)}\}}{1 - e^{-(2n+1)(\alpha_2-\alpha_1)} + 0\{1/n \; e^{-(n+1/2)(\alpha_2-\alpha_1)}\}}$$

which shows that

$$\underset{n \to \infty}{\text{Lt}} \quad u_n = 0.$$

Thus the necessary condition for the convergence of the series in (3.7) is satisfied.

$$\text{Now} \quad \underset{n \to \infty}{\text{Lt}} \left| \frac{u_{n+1}}{u_n} \right|$$

$$= \underset{n \to \infty}{\text{Lt}} \left| \frac{R_{n+1} \, (n+3/2) \, e^{-(n+3/2)(\alpha_2-\alpha)}}{R_n \, (n+1/2) \, e^{-(n+1/2)(\alpha_2-\alpha)}} \right|$$

$$= e^{-(\alpha_2-\alpha)} \underset{n \to \infty}{\text{Lt}} \left| \frac{R_{n+1}}{R_n} \right| < 1.$$

Thus the series of (3.7) is uniformaly convergent when $\alpha_1 < \alpha < \alpha_2$ and $0 \leqslant \beta \leqslant \pi$.

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