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# A NOTE ON FOX'S H-FUNCTION

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## ABSTRACT

In this note an attempt has been made to convert Fox's H-Function [3].

$$H_{p,q}^{m,n}\left[ z \,\middle|\, \begin{matrix} (a_1, e_1), \ldots\ldots\ldots, (a_p, e_p) \\ (b_1, f_1), \ldots\ldots\ldots, (b_q, f_q) \end{matrix} \right]$$

into Meijer's G-function [2, p. 207] when all e's and f's are rational numbers.

## 1. INTRODUCTION

The definition of the H-Function of Fox [3] will be denoted as [4]

$$(1.1) \qquad H_{p,q}^{m,n}\left[ x \,\middle|\, \begin{matrix} (a_1, e_1), \ldots\ldots\ldots, (a_p, e_p) \\ (b_1, f_1), \ldots\ldots\ldots, (b_q\ f_q) \end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - f_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + e_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + f_j s) \prod_{j=n+1}^{p} \Gamma(a_j - e_j s)} x^s \, ds.$$

where x is not equal to zero, and an empty product is interpreted as unity, $0 \leqslant m \leqslant q$ , $0 \leqslant n \leqslant p$ ; e's and f's are all positive integers, L is a suitable contour of Barnes type such that the poles of $\Gamma(b_j - f_j s)$, $j = 1, \ldots\ldots\ldots m$ lie on the right hand side and that of $\Gamma(1 - a_j + e_j s)$ $j = 1, \ldots\ldots n$, lie on the left hand side of the contour.

The asymptotic expansion and analytic continuum of the H-function is given in Braaksma [1].

For all e's and f's being equal to unity, we have the relation

$$(1.2)\quad H^{m,n}_{p,q}\left[ z \,\middle|\, \begin{matrix} (a_1, e_1), \ldots\ldots, (a_p, e_p). \\ (b_1, f_1), \ldots\ldots, (b_q, f_q) \end{matrix} \right]$$

$$= G^{m,n}_{p,q}\left[ z \,\middle|\, \begin{matrix} a_1, \ldots\ldots, a_p \\ b_1, \ldots\ldots, b_q \end{matrix} \right]$$

In the present note, we have attempted to convert Fox's H-Function defined above, into Meijer's G-function when all e's and f's are Rational numbers.

2. Let (i) $e_j = \dfrac{A_j}{B_j}$ for all $j = 1, 2, \ldots\ldots, p$

and (ii) $f_j = \dfrac{C_j}{D_j}$ for all $j = 1, 2, \ldots\ldots, q$

where all A's, B's, C's and D's are positive integers.

Consider the factors

$$(2.1)\quad \prod_{j=1}^{m} \Gamma(b_j - f_j s) = \prod_{j=1}^{m} \Gamma\left(b_j - \frac{C_j}{D_j} s\right)$$

$$= \prod_{j=1}^{m} \Gamma\left(b_j - c_j \prod_{i=1}^{q} D_i^{*} \prod_{k=1}^{p} B_k \, \xi\right)$$

where

$$(2\ 2)\quad s = \prod_{i=1}^{q} D_i \prod_{k=1}^{p} B_k \, \xi, \quad ds = \prod_{i=1}^{q} D_i \prod_{k=1}^{p} B_k \, d\xi$$

and $\prod_{i=1}^{q} D_i^{*}$ stands for $D_1, D_2 \ldots\ldots D_q$ in which the term $D_j$ do not present for every $j = 1, \ldots\ldots m$.

The term $C_j \prod_{i=1}^{q} D_i^{*} \prod_{k=1}^{p} B_k$ will be a positive integer since $C_j$, $D_i^{*}$ and $B_k$ are all integers for all i, j and k.

Let $C_j \prod_{i=1}^{q} D_i^{*} \prod_{k=1}^{p} B_k \equiv F_j$

The term of (2.1) will take the form

$$\prod_{j=1}^{m} \Gamma(b_j - F_j \xi).$$

Applying the multiplication formula for Gamma function viz.

$$(2.3) \quad \Gamma(\alpha + nz) = (2\pi)^{\frac{1-n}{2}} n^{\alpha + nz - \frac{1}{2}} \prod_{t=0}^{n-1} \Gamma\left(\frac{\alpha + t}{n} + z\right),$$

we have

$$(2.4) \quad \prod_{j=1}^{m} \Gamma(b_j - F_j \xi) = (2\pi)^{\left(m - \sum_1^m F_j\right)/2} \prod_{j=1}^{m} \left( (F_j)^{b_j - F_j - \frac{1}{2}} \right)$$

$$\times \prod_{j=1}^{m} \left( \prod_{t=0}^{F_j - 1} \Gamma\left( \frac{b_j + t}{F_j} - \xi \right) \right)$$

Similarly the term

$$(2.5) \quad \prod_{j=1}^{n} \Gamma(1 - a_j + e_j s) = \prod_{j=1}^{n} \Gamma\left(1 - a_j + \frac{A_j}{B_j} s\right)$$

$$= (2\pi)^{\left(n - \sum_1^n E_j\right)/2} \prod_{j=1}^{n} \left( (E_j)^{(1 - a_j + E_j \xi - \frac{1}{2})} \right)$$

$$\times \prod_{j=1}^{n} \prod_{t=0}^{E_j - 1} \Gamma\left( \left( \frac{1 - a_j + t}{E_j} + \xi \right) \right)$$

where

$$E_j = A_j \prod_{i=1}^{q} D_i \prod_{k=1}^{p} B_k^{*}$$

$\prod_{k=1}^{p} B_k^{*}$ will mean $B_1, B_2 \ldots\ldots B_p$ in which the term $B_j$ is not present for all $j = 1, \ldots\ldots n$.

Similar expressions for the terms $\prod_{j=m+1}^{q} \Gamma\,(1-b_j+f_j\,s)$

and $\prod_{j=n+1}^{p} \Gamma\,(a_j-e_j\,s)$ can easily be obtained.

Substituting these values and the value of s given in (2.2) in the expression (1.1), we have

(2.6) $$H_{p,\,q}^{m,\,n}\left[ z \,\middle|\, \begin{matrix} (a_1\, e_1), \ldots\ldots\ldots, (a_p, e_p) \\ (b_1, f_1) \ldots\ldots\ldots, (b_q, f_q) \end{matrix} \right]$$

$$= (2\pi)^{\alpha-\beta}\, \gamma\, \delta$$

$$\times \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \left( \prod_{t=0}^{F_j-1} \Gamma\left( \frac{b_j+t}{F_j} - \xi \right) \right) \prod_{j=1}^{n} \left( \prod_{t=0}^{E_j-1} \Gamma\left( \frac{1-a_j+t}{E_j} + \xi \right) \right)}{\prod_{j=m+1}^{q} \left( \prod_{t=0}^{F_j-1} \Gamma\left( \frac{1-b_j+t}{F_j} + \xi \right) \right) \prod_{j=n+1}^{p} \left( \prod_{t=0}^{E_j-1} \Gamma\left( \frac{a_j+t}{E_j} - \xi \right) \right)}$$

$$\times \left( \mathcal{J}\, z^{\delta} \right)^{\xi} d\,\xi$$

where $2(m+n)-(p+q)=\alpha$,

$$\sum_{j=1}^{m} F_j - \sum_{m+1}^{q} F_j + \sum_{1}^{n} E_j - \sum_{n+1}^{p} E_j = \beta,$$

$$\prod_{j=1}^{q} \left( F_j^{\,2b_j-1} \right) \prod_{j=1}^{p} \left( E_j^{\,1-2a_j} \right) = \gamma,$$

$$\prod_{i=1}^{q} D_i \prod_{k=1}^{p} B_k = \delta,$$

$$\prod_{j=1}^{p} \left( E_j^{\,2E_j\,\xi} \right) \Big/ \prod_{j=1}^{q} (F_j)^{2F_j\,\xi} = \mathcal{J}.$$

Interpreting the R. H. S. of (2.6) in view of [2, p. 207] we get the required result indicating the relation between H-function and G-function in case all e's and f's are rational numbers viz.,

$$(2.7) \quad H^{m,n}_{p,q}\left[ z \,\middle|\, \begin{matrix} (a_1, e_1), \ldots\ldots (a_p, e_p) \\ (b_1, f_1), \ldots\ldots (b_q, f_q) \end{matrix} \right]$$

$$= (2\pi)^{\alpha-\beta} \gamma \xi \; G^{\sum_{j=1}^{m} F_j, \; \sum_{j=1}^{n} E_j}_{\sum_{j=1}^{p} E_j, \; \sum_{j=1}^{q} F_j} \left[ \mathcal{J} z \,\middle|\, \begin{matrix} \Delta (E_p, a_p) \\ \Delta (F_q, b_q) \end{matrix} \right]$$

where $\triangle$ $(E_p, a_p)$ stands for $\triangle$ $(E_1, a_1), \ldots\ldots, \triangle (E_p\ a_p)$ and so for $\triangle$ $(F_q, b_q)$; and $\triangle$ $(k, \alpha)$ will stand for the set of k parameters

$$\frac{\alpha}{k}, \; \frac{\alpha+1}{k}, \ldots\ldots\ldots\ldots \frac{\alpha+k-1}{k}$$

**Particular case**

In particular, if all A's, B's, C's and D's be taken to be unity, in that case all $E_j$ $(j=1,\ldots\ldots p)$ and $F_j$, $(j=1,\ldots\ldots q)$ will be equal to unity, so we have the usual relation (1.2). In our subsequent communication, we will attempt to find out the possibility in case when all e's and f's of (1.1) are any real numbers.

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