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CERTAIN DOUBLE INTEGRALS INVOLVING HYPERGEOMETRIC FUNCTIONS*

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1. INTRODUCTION.

Making use of the familiar abbreviation

$$(1)\qquad (\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} = \begin{cases} 1, \text{ if } m=0, \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+m-1), \text{ if } m=1,2,3,\ldots, \end{cases}$$

we write

$$(2)\qquad {}_1F_1\ (\alpha\,;\gamma\,;z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m} \frac{z^m}{m!}\,,\ \gamma\neq 0,-1,-2,\ldots\ldots,$$

where ${}_1F_1\ (\alpha\,;\gamma\,;z)$ denotes Kummer's confluent hypergeometric function (cf., e.g., [1], p. 4).

The problem of evaluation of double integrals of the type

$$(3)\qquad \Delta = \int_0^{\pi}\int_0^{\pi} {}_1F_1\ (\alpha;\gamma\,;\lambda_1+\lambda_2\cos\psi+\lambda\cos\theta\sin\psi)\sin\psi\, d\psi\, d\theta,$$

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where the λ's are constants, arises in a study of the collision terms of the Boltzman equation in the kinetic theory of gases. Recently, Deshpande [2, p. 13] showed that, if $\mathrm{Re}(\alpha) > 1$ and $\mathrm{Re}(\gamma) > 1$, then

$$(4)\qquad \Delta = \frac{\pi}{R}\,\frac{\gamma-1}{\alpha-1}\left\{ {}_1F_1(\alpha-1;\gamma-1;\lambda_1+R) - {}_1F_1(\alpha-1;\gamma-1;\lambda_1-R) \right\},$$

where, for convenience,

$$(5)\qquad R^2 = \lambda^2 + \lambda_2^2.$$

In his long and involved proof of formula (4), Deshpande [loc. cit., pp. 11—13] made use of the contour integral [3, p. 272]

$$(6)\qquad {}_1F_1(\alpha;\gamma;z) = \frac{1}{2\pi i}\,\frac{\Gamma(\gamma)\,\Gamma(\alpha-\gamma+1)}{\Gamma(\alpha)} \int_0^{(1+)} e^{zt}\, t^{\alpha-1}\,(t-1)^{\lambda-\alpha-1}\,dt,\ \mathrm{Re}(\alpha) > 0,$$

the Neumann expansion [4, p. 98]

$$(7)\qquad \left(\frac{1}{2}z\right)^{\nu} = e^{-\gamma z}\,\Gamma(\nu) \sum_{n=0}^{\infty} (\nu+n)\, C_n^{\nu}(\gamma)\, I_{\nu+n}(z),$$

and the addition theorem for Legendre polynomials [5, p. 35]

$$(8)\qquad P_n(\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\varnothing) = P_n(\cos\alpha)\,P_n(\cos\beta) + 2\sum_{m=1}^{\infty} (-1)^m \frac{(n-m)!}{(n+m)!}\, P_n^m(\cos\alpha)\, P_n^m(\cos\beta)\cos m\varnothing,$$

where $C_n^{\nu}(x)$ denotes the egenbauer (or ultraspher.ca polynomial and

$$9)\qquad P_n^m(x) = (-2)^m \left(\frac{1}{2}\right)_m (1-x^2)^{m/2}\, C_{n-m}^{m+\frac{1}{2}}(\quad)$$

is an associated Legendre function (cf. also [4], p. 244).

The object of the present paper is to give a simple and direct method of evaluation of the double integral (3) without using the known formulas (7),

(8) and (9) and the orthogonality properties of the surface spherical harmonics involved. In §2 of this paper we prove formula (4) under the condition $\gamma-1\neq 0$, -1, -2,........, which evid is less restr tive than those required by the earlier writer [2] who assumes Re $(\alpha)>1$ and Re $(\gamma)>1$. The last section discusses a possible extension of the method, illustrated in §2, in order to evaluate more general definite integrals of the type

$$(10)\qquad \Omega=\int_0^{\pi}\int_0^{\pi} {}_{\nu}F_{\sigma}\left[\begin{matrix} a_1,\ldots\ldots, a_{\nu}; \\ b_1,\ldots\ldots, b_{\sigma}; \end{matrix}\ \lambda_1+\lambda_2\cos\psi+\lambda\cos\theta\sin\psi\right]\sin\psi\, d\psi\, d\theta,$$

where ${}_{\nu}F_{\sigma}[z]$ denotes the generalized hypergeometric function defined by [6, p. 275]

$$(11)\qquad {}_{\nu}F_{\sigma}\left[\begin{matrix} a_1,\ldots\ldots, a_{\nu}; \\ \\ b_1,\ldots\ldots, b_{\sigma}; \end{matrix}\ z\right]=\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{\nu}(a_j)_m}{\prod_{j=1}^{\sigma}(b_j)_m}\ \frac{z^m}{m!}.$$

For the usual restrictions on the b parameters and the conditions of convergence of the general series 11) see, for instance, Luke [7, pp. 43-44].

It may be of interest to remark in passing that a large variety of special functions are merely particular forms of the generalized hypergeometric ${}_{\nu}F_{\sigma}$ function involved in the double integral (10) above (see [7], pp. 209-225). Note, in particular, that by assigning special values to the parameters α and γ, with of course $\gamma-1\neq 0, -1, -2, \ldots\ldots$, the confluent hypergeometric ${}_1F_1$ function occurring on either side of formula (4) can be replaced by the Whittaker function $M_{k,\,m}(z)$, the Laguerre polynomial $L_n^{(\nu)}(z)$, the parabolic cylinder function $D_{\nu}(z)$, the Hermite function $H_{\nu}(z)$, the Bessel function $I_{\nu}(z)$, the incomplete Gamma function $\gamma(\nu, z)$, the error function Erf (z), and so on ([3], pp. 268-269; [6], pp. 271-274). Since these special functions occur frequently in various applied problems (cf., e. g., [8]), the definite integrals in (3) and (10) might find applications in areas other than the one that motivated the earlier writer [2] to evaluate the double integral (3).

For the sake of ready reference, we list here the following results that will be required in our analysis.

(12)
$$\sum_{m_1,\dots m_n=0}^{\infty} f(m_1+\dots+m_n)\frac{(x_1)^{m_1}}{m_1!}\cdots\cdots\frac{(x_n)^{m_n}}{m_n!}$$
$$=\sum_{m=0}^{\infty} f(m)\frac{(x_1+\dots\dots x_n)^m}{m!},\ n\geq 1,$$

provided that the series involved are absolutely convergent.

(13)
$$\int_0^{\pi}\cos^m t\ g(\sin t)\,dt$$
$$=\left\{1+(-1)^m\right\}\int_0^{\pi/2}\cos^m t\ g(\sin t)\,dt,\ m=0,1,2,\dots\dots\dots,$$

provided that the integrals exist.

(14)
$$\sum_{m=0}^{\infty} h(m)-\sum_{m=0}^{\infty}(-1)^m h(m)=2\sum_{m=0}^{\infty} h(2m+1),$$

provided that the series involved are convergent.

The identities (13) and (14) are not difficult to verify. On the other hand, the series relationship given by (12) can be proved fairly easily by induction. Indeed (12) holds true for $n=1$, and also for $n=2$, since

(15)
$$\sum_{m,n=0}^{\infty} f(m+n)\frac{x^m}{m!}\frac{y^n}{n!}=\sum_{N=0}^{\infty} f(N)\frac{y^N}{N!}\sum_{m=0}^{N}\frac{(-N)_m}{m!}\left(-\frac{x}{y}\right)^m$$
$$=\sum_{N=0}^{\infty} f(N)\frac{y^N}{N!}\left(1+\frac{x}{y}\right)^N=\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!}.$$

Now assuming that (12) remains valid for some value of the positive integer n, we notice that

$$\sum_{m_1,\ldots,m_{n+1}=0}^{\infty} f(m_1 + \ldots + m_{n+1}) \frac{(x_1)^{m_1}}{m_1!} \cdots\cdots \frac{(x_{n+1})^{m_{n+1}}}{m_{n+1}!}$$

$$= \sum_{m_{n+1}=0}^{\infty} \frac{(x_{n+1})^{m_{n+1}}}{m_{n+1}!} \sum_{m_1,\ldots,m_n=0}^{\infty} f(m_1 + \ldots + m_{n+1}) \frac{(x_1)^{m_1}}{m_1!} \cdots \frac{(x_n)^{m_n}}{m_n!}$$

$$= \sum_{m_{n+1}=0}^{\infty} \frac{(x_{n+1})^{m_{n+1}}}{m_{n+1}!} \sum_{m=0}^{\infty} f(m + m_{n+1}) \frac{(x_1 + \ldots + x_n)^m}{m!},$$

by using (12),

$$= \sum_{m,\, m_{n+1}=0}^{\infty} f(m + m_{n+1}) \frac{(x_1 + \ldots + x_n)^m}{m!} \frac{(x_{n+1})^{m_{n+1}}}{m_{n+1}!}$$

$$\sum_{M=0}^{\infty} f(M) \frac{(x_1 + \ldots + x_{n+1})^M}{M!}$$, by means of (15),

which shows that the identity (12) holds for n+1, if it is true for the positive integer n. This evidently completes the proof of (12) by the principle of finite mathematical induction.

We shall also require Legendre's duplication formula [3, p. 5]

(16) $$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\, \Gamma\left(z + \frac{1}{2}\right)$$

in the following more convenient forms :—

(17) $$(2m)! = 2^{2m}\, m! \left(\frac{1}{2}\right)_m,$$

(18) $$(2m+1)! = 2^{2m}\, m! \left(\frac{3}{2}\right)_m,$$

where m = 0, 1, 2,.........

2. PROOF OF FORMULA (4).

If we suppose that γ is neither zero nor a negative integer, then the confluent hypergeometric series given by (2) would satisfy the requirements of term-by-term integration over the region of integration of the double integral in (3). Making use of (2) and (12) in (3), we thus have

$$\Delta = \sum_{m,\, n,\, p=0}^{\infty} \frac{(\alpha)_{m+n+p}}{(\gamma)_{m+n+p}} \frac{(\lambda_1)^m}{m\,!} \frac{(\lambda_2)^n}{m\,!} \frac{(\lambda)^p}{p\,!}$$

$$\int_0^{\pi} \cos^n \psi \sin^{p+1} \psi \int_0^{\pi} \cos^p \theta \, d\theta,$$

which, in view of (13) and the familiar result [6, p. 14]

$$(19) \quad \int_0^{\pi/2} \cos^{\mu} t \sin^{\rho} t \, dt = \frac{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\rho+1}{2}\right)}{2\,\Gamma\left(\frac{\mu+\rho+2}{2}\right)}, \quad \mathrm{Re}\,(\mu) > -1,$$

$$\mathrm{Re}\,(\rho) > -1,$$

yields

$$(20) \quad \Delta = \sum_{m,\, q,\, r=0}^{\infty} \frac{(\alpha)_{m+2q+2r}}{(\gamma)_{m+2q+2r}} \frac{(\lambda_1)^m}{m\,!} \frac{(\lambda_2)^{q}}{(2q)\,!} \frac{(\lambda)^{2r}}{(2r)\,!} \frac{\sqrt{\pi}\,\Gamma\left(q+\frac{1}{2}\right)\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(q+r+\frac{3}{2}\right)}.$$

Now apply formula (17) on the right-hand side of (20) and put $q + r = s$. We find that

$$\Delta = 2\pi \sum_{m,\, s=0}^{\infty} \frac{(\alpha)_{m+2s}}{(\gamma)_{m+2s}\,(3/2)_s} \frac{(\lambda_1)^m}{m\,!} \frac{(\lambda/2)^{2s}}{s\,!} \sum_{q=0}^{s} \frac{(-s)_q}{q\,!} \left(-\frac{(\lambda_2)^2}{\lambda^2}\right)^q$$

$$= 2\pi \sum_{m,\, s\, =\, 0}^{\infty} \frac{(\alpha)_{m+2s}}{(\gamma)_{m+2s}\,(3/2)_s} \frac{(\lambda_1)^m}{m\,!} \frac{(\lambda/2)^{2s}}{s\,!} \left(1 + \frac{(\lambda_2)^2}{\lambda^2}\right)^s$$

$$= 2\pi \sum_{m,\, s\, =\, 0}^{\infty} \frac{(\alpha)_{m+2s}}{(\gamma)_{m+2s}\,(3/2)_s} \frac{(\lambda_1)^m}{m\,!} \frac{(R/2)^{2s}}{s\,!},$$

where R is given by (5).

In view of (1) and the formula (18), we have

$$\Delta = 2\pi \sum_{m,\, s=0}^{\infty} \frac{(a)_{m+2s}}{(\gamma)_{m+2s}} \frac{(\lambda_1)^m}{m\,!} \frac{R^{2s}}{(2s+1)\,!}$$

$$= \frac{2\pi}{R} \frac{\gamma - 1}{a - 1} \sum_{m,\, s\, =\, 0}^{\infty} \frac{(a-1)_{m+2s+1}}{(\gamma-1)_{m+2s+1}} \frac{(\lambda_1)^m}{m\,!} \frac{R^{2s+1}}{(2s+1)\,!},$$

and by an appeal to the identities (14) and (15), we finally have

$$\Delta = \frac{\pi}{R} \frac{\gamma - 1}{a - 1} \left\{ \sum_{m,\, s\, =\, 0}^{\infty} \frac{(a-1)_{m+s}}{(\gamma-1)_{m+s}} \frac{(\lambda_1)_m}{m\,!} \frac{R^s}{s\,!} - \sum_{m,\, s\, =\, 0}^{\infty} \frac{(a-1)_{m+s}}{(\gamma-1)_{m+s}} \frac{(\lambda_1)^m}{m\,!} \frac{(-R)^s}{s\,!} \right\}$$

$$= \frac{\pi}{R} \frac{\gamma - 1}{\alpha - 1} \left\{ {}_1F_1\,(a-1;\gamma-1;\, \lambda_1+R) - {}_1F_1\,(a-1;\, \gamma-1;\, \lambda_1-R) \right\},$$

provided, by the principle of analytic continuation,

(21) $\qquad \gamma - 1 \neq 0, -1, -2, \ldots\ldots\ldots$.

This completes the proof of formula (4) under the condition given by equation (21).

3. EVALUATION OF THE DOUBLE INTEGRAL (10).

We now turn to the problem of evaluation of the definite integral in (10). Indeed if, for convergence of the infinite series involved, we let $\nu \leqq \sigma$ (or $\nu=\sigma+1$ and $|\lambda_1|+|\lambda_2|+|\lambda|<1$) and suppose that none of the b parameters is a negative integer or zero, then the method illustrated in the preceding section can be applied equally well to the double integral (10), and we shall readily get

$$\Omega = 2\pi \sum_{m,\,s=0}^{\infty} \frac{\prod_{j=1}^{\nu} (a_j)_{m+2s}}{\prod_{j=1}^{\sigma} (b_j)_{m+2s}} \frac{(\lambda_1)^m}{m\,!} \frac{R^{2s}}{(2s+1)\,!}$$

$$= \frac{2\pi}{R} \frac{\prod_{j=1}^{\sigma} (b_j-1)}{\prod_{j=1}^{\nu} (a_j-1)} \sum_{m,s=0}^{\infty} \frac{\prod_{j=1}^{\nu} (a_j-1)_{m+2s+1}}{\prod_{j=1}^{\sigma} (b_j-1)_{m+2s+1}} \frac{(\lambda_1)^m}{m\,!} \frac{R^{2s+1}}{(2s+1)\,!}.$$

Applying the identities (14) and (15), we finally have

$$\Omega = \int_0^{\pi} \int_0^{\pi} {}_{\nu}F_{\sigma} \left[\begin{matrix} a_1, \ldots\ldots\ldots\ldots, a_{\nu}\, ; \\ b_1, \ldots\ldots\ldots\ldots, b_{\sigma}\, ; \end{matrix} \; \lambda_1+\lambda_2 \cos\psi+\lambda \cos\theta \sin\psi \right] \sin\psi \, d\psi \, d\theta$$

$$(22) \quad = \frac{\pi}{R} \frac{\prod_{j=1}^{\sigma} (b_j-1)}{\prod_{j=1}^{\nu} (a_j-1)} \left\{ {}_{\nu}F_{\sigma} \left[\begin{matrix} a_1-1, \ldots, a_{\nu}-1\, ; \\ b_1-1, \ldots, b_{\sigma}-1\, ; \end{matrix} \; \lambda_1+R \right] - {}_{\nu}F_{\sigma} \left[\begin{matrix} a_1-1, \ldots\ldots, a_{\nu}-1\, ; \\ b_1-1, \ldots\ldots, b_{\sigma}-1 \end{matrix} \; \lambda_1-R \right] \right\},$$

where, as before, R is given by (5) and, by analytic continuation,

$\nu \leq \sigma$ (or $\nu = \sigma + 1$ and $\max \left\{ |\lambda_1| + |\lambda_2| + |\lambda|, |\lambda_1 + R| \right\} < 1$) and

(23) $b_j - 1 \neq 0, -1, -2, \ldots\ldots; j = 1, \ldots\ldots, \sigma$.

In case, however, one or more of the a parameters are zero or a negative integer, the hypergeometric series occuring in the last formula (22) would terminate, and the aforementioned restrictions on ν, σ and the λ's can be waived.

Evidently formula (22), with $\nu = \sigma = 1$, $a_1 = a$ and $b_1 = \gamma$, would reduce to the earlier result given by equation (4).

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