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COLLINEATION INDUCED INCIDENCE RELATIONS*

By

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1. INTRODUCTION

The triangle has associated with it a number of wellknown remarkable points. While each of these possess Individual properties important in the geometry of the triangle, certain subsets share the further property of lying on the same line . Some of these, such as the Euler line which contains the centroid, circumcentre, orthocentre, and nine-point centre of the given triangle, have been known for some time while others, described later, are more recent. In this paper, we apply simple collineations to aome known incidence relations and obtain others believed to be new . Interestingly, all of the latter except one, involve a seemingly little known point, the "middlespoint" that appears in a paper by C. H. von Nagel published in 1836.

As a secondary objective, we attempt to demonstrate how classical Euclidean geometry of the plane may be enriched and made more interesting by the use of other techniques, in this case, those of projective geometry, In return, material that is essentially Euclidean, but which has a natural link with real projective geometry, can help make more concrete, and hence more appealing, the abstract area of general projective geometry. Since the latter, with its attention to

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rigor, generally takes precedence over the Euclidean in present-day curricula, we shall emphasize the second interaction.

2. Background.

In the Euclidean plane E^2, any two points determine a unique line. This also holds in the projective plane P^2 where we have, additionally, the dual statement every pair of lines determine a unique point and parallelism is not an issue. While this more general setting can not be handled by the usual Cartesian coordinate system, a related system works quite nicely.

Consider the equations

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3}$$

where x_i, $i = 1, 2, 3$, are real and $x_3 \neq 0$, then each ordered triple (x_1, x_2, x_3) has the ordered pair (x, y) as a unique mate in E^2 when only the ratios $x_1 : x_2 : x_3 = x : y : 1$ are considered. The triple (x_1, x_2, x_3) thus represents the homogeneous coordinates of a point p in the projective plane. Since all non-zero scalar multiples of a given triple represent the same point, we have the added convenience of working with integer coordinates. For example, the mate of the point $\left(\frac{1}{2}, \frac{-2}{7}\right)$ in E^2 has coordinates $\left(\frac{1}{2}, \frac{-2}{7}, 1\right) = (7, -4, 14)$ in p^2.
The points $(x_1, x_2, 0)$ have no mates in E^2 and thus constitute the line at infinity, $l_\infty : x_3 = 0$, in p^2.

There is, in general, on projective counterpart for the Cartesian distance formula thus projective geometry with such non-metric notions as straightness and tangency One can include parallelism in this non-metric framework by singling out the line at infinity and defining a pencil of parallel lines as one with its centre on this line. This geometry, which is sort of in between Euclidean and projective, is called affine geometry.

A fundamental property of the projective plane is that it has a collineation structure, *i.e.* there exists a map called a collineation, that preserves straight lines (point ranges) and line pencils. This map, defined by the rule

$$\rho x'_i = \sum_j a_{ij} x_j \, ; \, i, \ j = 1, 2, 3, \qquad (1)$$

where the determinant $| a_{ij} | \neq 0$ and ρ is a non-zero constant, maps collinear points to image sets which are also collinear . when dealing with the geometry of the triangle, one needs to use the measures of the sides and angles of the given triangle. Two systems of specialized homogeneous coordinates that do this, while at the same time retain all the advantages of the general homogeneous type, are the trilinear (or normal) system and the barycentric (or areal) system.

We now provide a brief description of each of these systems, see [5] or [7]. In the trilinear case, the coordinates of a point $P\,(x_1, x_2, x_3)$ in the plane are proportional to the signed distances d_1, d_2, d_3 of P from the sides of a given triangle of reference $A_1A_2A_3$. The vertices of this triangle have coordinates representations $A_1\,(1, 0, 0)$, $A_2\,(0, 1, 0)$, $A_3\,(0, 0, 1)$ and the unit point of the system is the incentre $I=(r,r,r) = (1,1,1)$, where r is the inradius of $A_1A_2A_3$. Tee distance d_i, $i=1,2,3$ is positive if p is in the half-plane determined by I and A_jA_k, $i \notin \{ j, k \}$, and negative otherwise. The excentre I_1 opposite vertex A_1, for example, has coordinates $(-1, 1, 1)$ or its projective equivalent $(1, -1, -1)$. In the barycentric case, the coordinates of $P\,(y_1, y_2, y_3)$ are proportional to the signed areas of the triangle $PA_2A_3, PA_3A_1, PA_1A_2$ with the signs being determined as in the trilinear case with the unit point of the system being the centroid S. The line at infintiy is the unit line $x_1 + x_2 + x_3 = 0$, while for the trilinear case it is $a_1x_1 + a_2x_2 + a_3x_3 = 0$, where a_i, $i = 1, 2, 3$ are the measures of the sides of the reference triangle. We shall use this system for the remainder of the paper.

Incidence relations are readily determined by such a system. Consider the points $P_1(x_1, x_2, x_3)$, $P_2(y_1, y_2, y_3)$, $P_3(z_1, z_2, z_3)$ respectively; then P_1, P_2, P_3 are collinear if and only if the determinantal equation

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0 \tag{2}$$

is true.

For concurrency of lines, there exists a dual procedure. If, for example, we represent the line $l_1 : u_1x_1 + u_2x_2 + u_3x_3 = 0$ by its "line" or "pluckerian" coordinates $l_1(u_1, u_2, u_3)$. see [6], then l_1, $l_2(v_1, v_2, v_3)$, and $l_3(w_1, w_2, w_3)$ are concurrent if and only if

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0\,. \tag{3}$$

In Table 1, we list several of the special points in the plane associated with the triangle of reference $A_1A_2A_3$ along with their coordinate representations.

Table 1

Point	Symbol	Trilinear Coordinatea
Incentre	I	$(1, 1, 1)$
Excentre	I_{A_1}	$(-1, 1, 1)$
Centroid	S	$\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}\right)$
Circumcentre	O	$(\cos \alpha_1, \cos \alpha_2, \cos \alpha_3)$
Orthocentre	H	$\left(\frac{1}{\cos \alpha_1}, \frac{1}{\cos \alpha_2}, \frac{1}{\cos \alpha_3}\right)$
Lemoine point	K	(a_1, a_2, a_3)
Nagel point	G	$\left(\frac{s-a_1}{a_1}, \frac{s-a_2}{a_2}, \frac{s-a_3}{a_2}\right)$, $s = \frac{a+b+c}{2}$
Gergonne point	N	$\left(\frac{1}{a_1(s-a_1)}, \frac{1}{a_2(s-a_2)}, \frac{1}{a_3(s-a_3)}\right)$
Middlespoint	M	$(s-a_1, s-a_2, s-a_3)$

The derivation of these coordinates is especially easy when the segments, which the concurrent lines through the given point and the vertices of $A_1A_2A_3$ cut the opposite sides, are known. Consider the point P and the pencil of lines A_1PD_1, A_2PD_2, A_3PD_3, where D_1 is on A_2A_3, etc., see figure 1, then for the point

D_1, $d_2 : d_3 = \frac{\overline{D_1A_3} \sin\alpha_3}{\overline{A_2D_1} \sin\alpha_2}$, where $\alpha_1, \alpha_2, \alpha_3$ denote the measures of the angles at A_1, A_2, A_3 respectively, see Table 1. Using the well known expressions $a_2 = 2R\sin\alpha_2$ and $a_3 = 2R\sin\alpha_3$, where R is the circumradius of $A_1A_2A_3$, which relate the measures of the sides and angles of the given triangle, we obtain

$$D_1 = (0, a_3 \overline{D_1 A_3}, a_2 \overline{A_2 D_1}) \tag{4}$$

as the trilinear coordinates of D_1. Since the coordinates for D_2 and D_3 may be derived in a similar manner, the coordinates of P are immediate.

To illustrate the idea, we consider three points from table 1 that might not be widely known and indicate briefly how to derive their eorresponding coordinates All three are readily derived as above since the positions of the D_i, $i=1, 2, 3$, in each case are known. We first consider the Lemoine or symmedium point K where the D_i's are such that

$$\frac{A_j D_i}{D_i A_k} = \frac{a_k^2}{a_j^2}, \quad i, j, k = 1.2, 3.$$

In this case, $D_1 = (0, a_2, a_3)$ with similar representations for D_2 and D_3, hence $K = (a_1, a_2, a_3)$.

Also fitting this picture are the Gergonne and Nagel points G and N shown in figures 2 and 3 respectively.

The coordinates of each of these points now follow readily since for G, $\overline{A_2 D_1} = s - a_2$ and for N, $\overline{A_2 D_1} = s - a_3$, etc.

We now come to the last point in table 1 and one which plays a key role in this paper, see figure 4.

Let S_1, S_2, S_3 denote the midpoints of $\overline{A_2 A_3}$, $\overline{A_3 A_1}$, $\overline{A_1 A_2}$ respectively, then the lines $I_1 S_1$, $I_2 S_2$, $I_3 S_3$ are concurrent at the middlespoint (mittenpunkt), N of the given triangle, see Nagel [4].

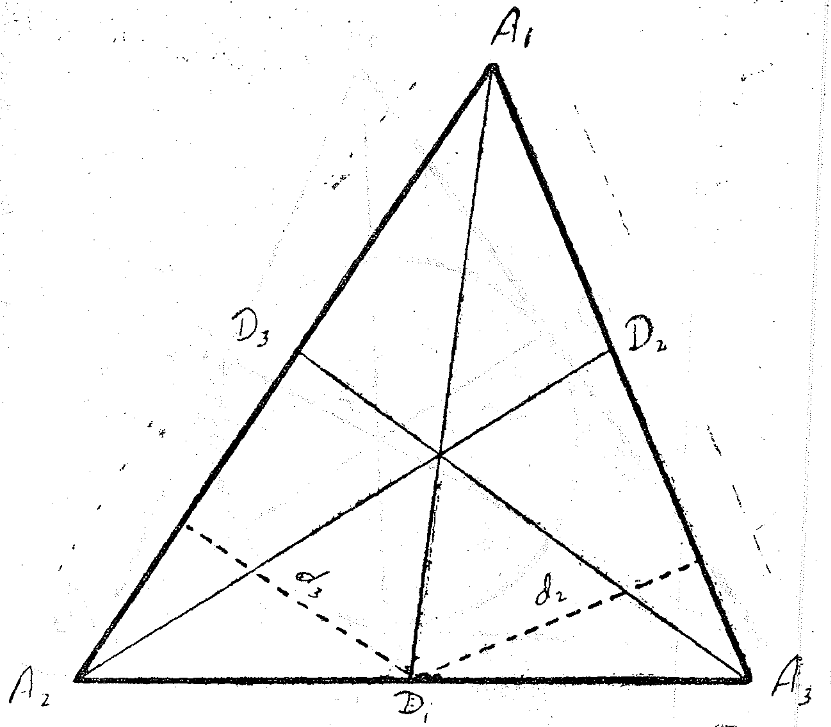

FIGURE 1

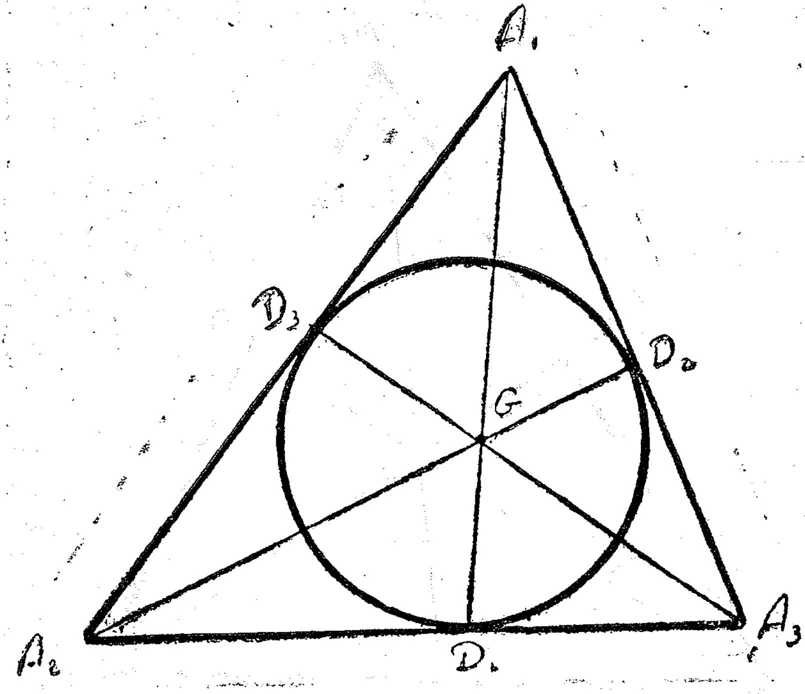

FIGURE 2

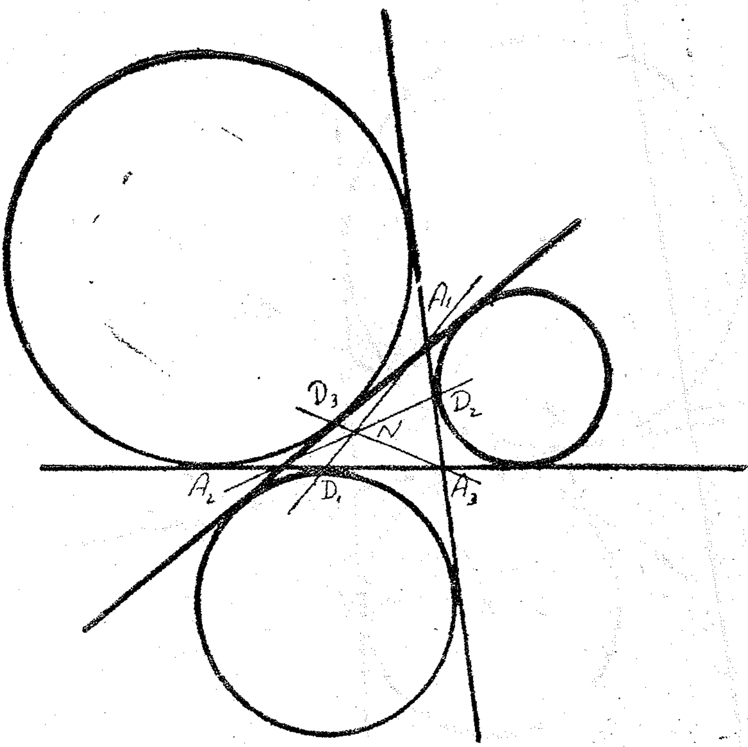

FIGURE 3

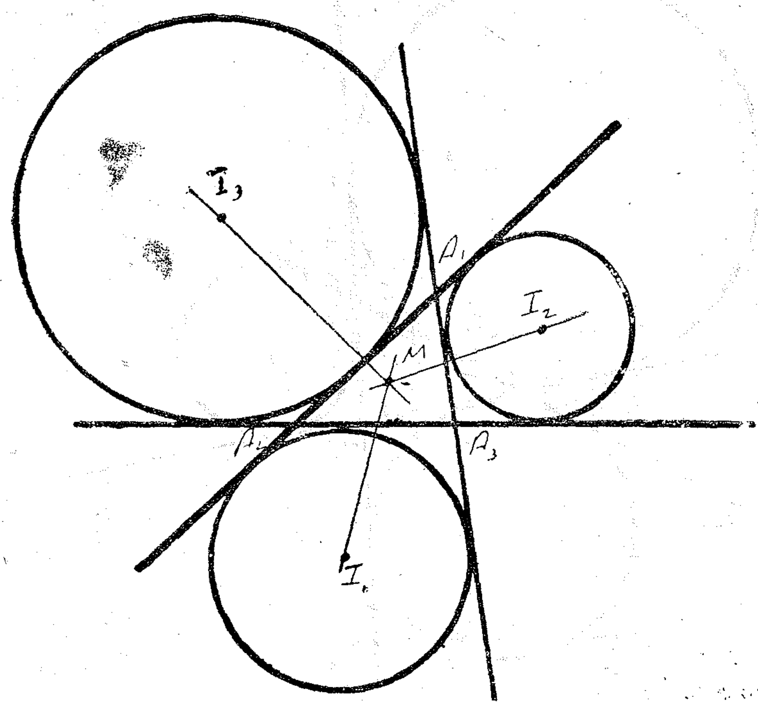

FIGURE 4

Again, the derivation of the coordinates is straightforward. The lines I_1S_1, I_2S_2, I_3S_3 are easily seen to have pluckerian coordinates $(a_2-a_3, a_2, -a_3)$, $(-a_1, a_3-a_1, a_3)$, $(a_1, -a_2, a_1-a_2)$, consequently, the concurrency property follows immediately by (3). Eor the coordinates of M, consider the determinantal equation

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ a_2-a_3 & a_2 & -a_3 \\ -a_1 & a_3-a_1 & a_3 \end{vmatrix} = 0, \qquad (5)$$

where u_1, u_2, u_3 are the coordinates of a general line in the plane.

Expanding, we obtain the pencil.

$$(a_2 + a_3 - a_1)\, u_1 + (a_2 + a_1 - a_2)\, u_2 + (a_1 + a_2 - a_3)\, u_3 = 0 \qquad (6)$$

with centre $M = (s - a_1, s - a_2, s - a_3)$. This interesting point, which perhaps derives it name from the fact that its construction involves "middles" *i. e. centres* of circles and the *midpoints* of line segments, seems to have been discovered by Nagel and, surprisingly, does not appear to have found its way into the popular literature dealing with the geometry of the triangle While some points may be "born to blush unseen" either because of a complicated construction or a cumbersome coordinate representation, this is certainly not the case with the middlespoint. The only other paper known to us that deals with this point is a paper by Baptist [1], soon to appear, which contains, in particular, proposition 1 below.

Other special points may be obtained from the known ones by applying two quadratic transformations. As will be seen, these turn out to be involutions that are well-defined for all poiuts in the plane except those incident with the triangle of reference, The firsi of these is the "isogonal" transformation defined as follows.

If the lines $A_i D_i$, $i=1, 2, 3$, through P are each reflected in the angle bisector through the same vertex, the reflected lines $A_i D_i'$ are concurreni at P^g, ihe isogonal conjugate of P, see [3].

It is an immediate consequence of the definition thati f P has trilinear coordinates (x_1, x_2, x_3), then P^g has coordinates

$$\left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3} \right) = (x_2 x_3, x_3 x_1, x_1 x_2) .$$ Two such sets, immediately seen from Table 1, are $\{O,H\}$ and $\{S, K\}$.

The second transformation is called "isotomic" and is defined as in the following manner.

If the points D_i' are the reflections of D_i in the midpoints S_i of the sides of the triangles $A_1A_2A_3$, the lines A_iD_i' concur at P^t, the isotomic conjugate of P, see [3].

Again, it is straightforward to ascertain that the trilinear coordinates of P^t are $\left(\frac{1}{a_1^2 x_1} \cdot \frac{1}{a_2^2 x_2}, \frac{1}{a_3^2 x_3} \right)$. It is readily seen from table 1 that $\{ G, N \}$ is a pair of isotomic conjugates.

3. **The results.** We are now in a position to derive results believed to be new. As indicated in the title, these, will be obtained by applying certain collineations to known incidence relations stated in the form of propositions. Each of the collineations is defined by a matrix $B = (b_{ij})$; $i, j = 1, 2, 3$, which is diagonal with $b_{11} . b_{22}$, $b_{33} \neq 0$. In general, this olsss of transformation forms a group

which fixes the triangle of reference and is sharply transitive on the points of the plane not incident with its sides. Thus, for convenience we represent each collineation by the general diagonal element b_{ii}, $i = 1, 2, 3$ of its defining matrix.

PROPOSITION 1. The incentre, Nagel, point, and the centroid of a triangle are collinear, see [3].

If we transform this set by the matrix $b_{ii} = a_i$, we obtain the points (a_1, a_2, a_3), $(s - a_1, s - a_2, s - a_3)$, $(1, 1, 1)$. Thus, we have derived

RESULT 1. The Lemoine point, the middlespoint and the incentre of a tainngle are collinear.

PROPOSITION 2. The isogonal conjugate os the Gergonne point, and the isogonal conjugate of the Nagel point, and the incentre are collinear, see [2].

Here we apply the collineation $b_{ii} = \dfrac{1}{a_i}$ and obtain the points

$$\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}\right), \ (s-a_1, s-a_2, s-a_3), \left(\frac{1}{s-a_1}, \frac{1}{s-a_2}, \frac{1}{s-a_3}\right).$$

We now have

RESULT 2. The middlespoint of a given triangle is collinear with its isogonal conjugate and the centroid.

If we consider again the points of proposition 2 and apply the collineation $b_{ii} = \dfrac{1}{a_i^2}$ we obiain the points $\left(\dfrac{1}{a_1^2}, \dfrac{1}{a_2^2}, \dfrac{1}{a_3^2}\right)$,

$$\left(\frac{s-a_1}{a_1}, \frac{s-a_2}{a_2}, \frac{s-a_3}{a_3} \right), \left(\frac{1}{a_1 (s-a_1)}, \frac{1}{a_2 (s-a_2)}; \frac{1}{a_3 (s-a_3)} \right.$$

thus we have

RESULT 3. The isotomic conjugate of the incentre, the Nagel point, and the Gergonne point are collinear.

PROPOSITION 3. The Gergonne point, the middlespoint, and the centroid of a triangle are collinear, see [1].

We now apply the same collineation as for result 2 and obtain the points

$$\left(\frac{1}{a_1^2 (s-a_1)}, \frac{1}{a_2^2 (s-a_2)}, \frac{1}{a_3^2 (s-a_3)} \right), \left(\frac{s-a_1}{a_1}, \frac{s-a_2}{a_2}, \frac{s-a_3}{a_3} \right),$$

$\frac{1}{a_1^2}, \frac{1}{a_2^2}, \frac{1}{a_2^2})$. we can now state

RESULT 4. The isotomic conjugate of the middlespoint, the Nagel point, and the isotomic conjugate of the incentre are collinear.

We now summarize these results in Table 2. For brevity, the collineations are again represented by the general diagonal terms of the defining matrices.

Table 2

	Known Results		Derived Results
1.	$\{I, N, S\}$	$\xrightarrow{a_i}$	$\{K, M, I)$
2.	$\{G^g, N^g, I\}$	$\xrightarrow{\frac{1}{a}}$	$\{M, M^g, S\}$
3.	$\{G^g, N^g, I\}$	$\xrightarrow{\frac{1}{a_i^2}}$	$\{I^t, N, G\}$
4,	$\{G, M, S\}$	$\xrightarrow{\frac{1}{a_i}}$	$\{M^t, N, I^t\}$

By examining the results in the table, we are able to state two further results. From the incident sets $\{M, M^g, S\}$ and $\{G, M, S\}$, we obtain

RESULT 5. The middlespoint of a triangle and its isogonal conjugate nre collinear with the Gergonne point and the centroid.

Similarly, thr sets $\{I^t, N, G\}$ and $\{M^t, N, I^t\}$ yield.

RESULT 6. The Nagel and Gergonne points of a triangle are collinear with the isotomic conjugates of the incentre and the middlespoint .

4. Conclusion As indicated earlier, material of this sort has a twofold value, Not only does it provide concrete examples that may be used in teaching the more abstract projective geometry but also, it helps to keep alive at least a portion of that all too quickly fading subject of Euclidean gcometry. The material itself should fit the curriculum of most undergraduate courses in geometry and since, many other such relations no doubt exist, there is good potential here for student research projects. Since, in our view, there is no greatcr motivator than the joy of discovery, such activity is a necessary part of the teaching-learning process. As a concluding remark we note that while much has been written, so also much nemains to be discovered about that most fascinating configuration–the triangle .

REFERENCES

[1] P. Baptist, Uber Nagelsche punkteparre, *Mathematische Semesterberichte* (To apear).

[2] J. T. Groenman, Problem 603, *Nieuw Archief voor Wiskunde.* XXIX 3 (1981) .

[3] R. A. Johnson, *Advanced Euclidean Geometry* (*Modern Geometry*), Dover, NY (1969) .

[4] C. H. V. Nagel, *Untcrsuchungen uber die wichtigsten zum Dreiecke gehorenden Kreise*, Leipzig (1836) .

[5] D. M. Y. Sommerville, *Analytical Conics,* G. Bell and Sons Ltd., London (1956) .

[6] C. E. Springer, *Geometry and Analysis of Projective Spaces*, W. H. Freeman, San Francisco (1964) .

[7] D. J. Struik, *Lectures on Analytic and Projective Geometry*, Addison-Wesley, Reading Mass. (1953) .