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# CAPILLARY INSTABILITY OF A STREAMING HOLLOW CYLINDER

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## ABSTRACT

The capillary instability of a hollow jet flow of a streaming liquid surrounding a gas cylinder ( the equilibrium gas pressure is held constant ) subject to its liquid inertia force is presented. A general eigenvalue relation is derived and studied analytically, and the results are confirmed numerically. A comparison is made between the present results and those for the nonhollow-cylinder. The streaming has always a destabilizing influence. The latter attains its maximum value if the basic flow streams in the opposite $z$-direction. In the absence of streaming, the model is stable for all non-axisymmetric modes for all wavelengths while to axisymmtric mode it is stable ( or not ) if the perturbed wavelength is equal to or shorter ( or longer ) than the circumference of the cylinder. The maximum temporal amplification prevailing in the nonhollow case is far lower than of the present model ; this is physically interpreted here.

## 1. INTRODUCTION

Crucial interest in the linear ( [6], [8], [1] ) and nonlinear ( [10], [4, [5], [2] ) hydrodynamic instability of cylindrical jets has been revised due to certain astronomical and industrial problems for many applications ranging from the design of sprays to the design of ink jet printers.

In recent years studies on the hollow cylinder stability have been flourishing. In view of its practical applications in the astrophysics science ; Kendall [3] has performed interesting experiments with modern equipment to the annular jet ( gas cylinder submerged in liquid jet ) under the action of surface tension and inertia forces provided that the inertia liquid force is paramount over that of the gas cylinder. Moreover, Kandell [3[ called attention to analytically studying the stability of such models. He mentioned : "it appears that the hollow jet instability and enscapsulation process being studied here have not been repeated in the literature" ( see [3. p. 2086] ).

The purpose of the present work is to investigate the instability of a streaming hollow cylinder endowed with surface tension and acting upon its liquid inertia force. It is worthwhile to mention here that the phenomenon of the hollow cylinder may occur in the crust of the earth when a gas escaping from below oil layers or in the sea during geological drillings or when air is pumped into the water.

## 2. FORMULATION OF THE PROBLEM AND EIGENVALUE RELATION

Let us consider a uniform gas cylinder of radius $r=R_0$. The cylinder is immersed in an infinite liquid of density $\rho$. The liquid has a basic flow of uniform velocity $U$ along the axis of the hollow cylinder. The model ( hollow jet ) is acting upon the capillary ( along the gas-liquid interface ) and inertia forces such that the liquid inertia force is paramount [3] over that of the gas. The effect of gravity is negligible for ( $gR_0/U^2$ ) $<<1$. We shall use the cylindrical polar coordinates $(r,\rho,z)$ with the axis of the hollow cylinder coinciding with the z-axis. The fluids are considered to be inviscid and incompressible. The basic equations are those of motion, continuity and that one of surface tension viz [1]

(1) $$\rho \frac{d\underline{u}}{dt} = \rho \left( \frac{\partial \underline{u}}{\partial t} + ( \underline{u} \cdot \nabla ) \underline{u} \right) = - \nabla p$$

(2) $$div\ \underline{u} = 0$$

(3) $$P_s = - T ( r_1^{-1} + r_2^{-1} ) = - T\, div\ \underline{n}_s$$

where $\underline{u}$ $P$ and $T$ are the liquid velocity, pressure and surface tension coefficient : $r_1$ and $r_2$ are principle radii of curvature of the cylindrical surface and $\underline{n}_s$ ( $= \nabla f / | \nabla f |$ ) is a unit outward vector normal ( directed from the gas to the liquid ) to the surface $f( r,\rho,z ; t ) = 0$.

The equilibrium pressure $P_0$ is given by

(4) $$P_0 = P_g - ( T/R_0 ) .$$

The term ( $- T/R_0$ ) is the surface tension contribution and $P_g$ is a gas constant pressure. The hollow cylinder has to be filled with gas of strong ( constant ) pressure to insure the equilibrium state snch that $P_g \geqslant T/R_0$ ( where equality may occur as limiting case with zero liquid pressure ) otherwise the model will collapse towards a hollow cylinder of smaller radius than $R_0$ to reach an equilibrium state.

For small departures from the equilibrium state, let the small departure in the radius of the cylinder be $\zeta$ and proportional to exp [ $i ( \alpha z + m\rho ) - i\alpha c t$ ] where $\zeta << R_0$.

(5) $$r = R_0 + \zeta = R_0 + \varepsilon_0 R_0\, exp [ i ( \alpha z + m \rho ) - i \alpha c t ]$$

where $\varepsilon_0$ is the initial amplitude ( at time $t = 0$ ) ; $\alpha$ and $m$ are the longitudinal and azimuthal wavenumbers, $c$ is the velocity of light. $\zeta$ represented the surface wave elevation normalized with respect to $R_0$ and measured from the equilibrium position. The linearized equations are

(6) $$\rho \left( \frac{\partial \underline{u}_1}{\partial t} + U \frac{\partial \underline{u}_1}{\partial z} \right) = - \nabla P_1$$

(7) $$div\, \underline{u}_1 = 0$$

(8) $$P_{1s} = ( T/R_0^2 ) ( \zeta = ( \partial^2\zeta / \partial\rho^2 ) + R_0^2 ( \partial^2\zeta / \partial z^2 ) ) : \text{ at } r = R_0$$

Consequent to deformation (5) as usual for the stability problems of cvlindrical configurations, based on the linear-perturbation technique we assume that any perturbed quantity can be expressed as $[ i( \alpha z + m\rho ) - i\alpha\, c\, t ]$ times an amplitude function of $r$. The relevant perturbation equations are solved and the non-singular soiution is given by

(9) $$\underline{u}_1 = A [ \alpha K'_m ( \alpha r ), \frac{im}{r} K_m ( \alpha r ), i\alpha K_m ( \alpha r ) ]$$

$$exp [ i ( \alpha z + m\rho ) - i\, \alpha\, c\, t ]$$

(10) $$P_1 = - \rho ( U - c ) u_{1z}$$

(11) $$\varepsilon_0^{-i} P_{1s} = ( T/R_0 ) ( 1 - m^2 - \alpha^2 R_0^2 ) exp [ i ( \alpha z + m\rho ) - i\alpha c t ],$$

where A is unspecifed constant and $K_m ( \alpha r )$ is the modified Bessel function of the second kind of the order $m$. The appropriate boundary conditions are : (I) the kinematic condition that the normal component of the fluid velocity must be compatible with the deformed surface (5) at $r = R_0$, yields $u_{1r} = \partial\zeta/\partial t + U\,\partial\zeta/\partial z$ from which one gets $A = iR_0\,(U - c)\,/\,K'_m\,(\alpha R_0)$ ; and (II) the kinetic pressure of the fluid must cope with the pressure due to the capillary force at $r = R_0$ . The last condition gives, at once, the following relation :

$$(12)\quad \omega^2 = (T/\rho R_0^3)(x^2 + m^2 - 1)\,F_m(x)$$

where $\omega = (U - c)\,\alpha$

$$F_m(x) = -x\,K'_m(x)\,/\,K_m(x)$$

and where $x\ (= \alpha R_0)$ is the longitudinal dimensionless wavenumber.

## 3. DISCUSSIONS AND CONCLUSIONS

Equation (12) is the desired eigenvalue relation of a capillary streaming hollow cylinder ( valid for all modes of perturbation $m \geqslant 0$ ) subjecting to its liquid inertia force . It relates the angular frequency $\omega$ ( or rather the temporal amplification $\sigma$ if $\omega$ is imaginary ( $\sigma = -i\alpha c$ ) ) with the entity $(T/\rho R_0^3)^{-1/2}$ as unit of time, the wave-number $x$ and $m$ , the second kind modified Bessel function and its derivative ( with respect to its argument ), the characteristic length $R_0$ and the other parameters $T$, $\rho$, $U$ and $c$ of the problem .

In the absence of streaming ( $U = 0$ ) and to axisymmetric disturbances $m = 0$ ; equation (12) reduces to

$$(13)\quad \sigma^2 = (T/\rho R_0^2\ [\,xK_1(x)\,/\,K_0(x)\,]\,(x^2 - 1)\,.$$

This coincides with dispersion relation obtained by us [7] as the viscosity effect is suppressed there .

By using the recurrence relation of the modified Bessel function

$$(14) \quad 2 K'_m (x) = - K_{m-1} (x) - K_{m+1} (x)$$

and to each non-zero real value of $x$ such that $K_m (x)$ is monotonic decreasing but never negative . One can show that

$$(15) \quad F_m (x) > 0$$

to all modes of perturbation $m \geqslant 0$ for all ( long and short ) wavelengths and that $F_m (x)$ never changes sign . By using the inequality (15); equation (12) yields that $\omega^2 > 0$ for all $m \neq 0$, but that $\omega^2 < 0$ for $-1 < x < 1$ and $\omega^2 \geqslant 0$ for $x \geqslant 1$ or $x \leqslant -1$ if $m = 0$ . That is exactly the same situation for the nonhollow cylinder (full liquid jet) . It is stable to all nonaxisymmetric disturbances $m \geqslant 1$ for all wavelengths while to $m=0$ it is stable ( or not ) if the perturbed wavelength is equal to or shorter ( or longer ) than the circumference of the jet. From the analytical discussions of eq. (12), one can easily show that the streaming has a destabilizing effect . The latter attains its maximum value if the basic flow streams (for given $U$) in the opposite $z$-direction . The destabilizing character of the streaming is also true even in the presence of different forces [9] . The analytical results are confirmed numerically by using the relation (12) and that of full liquid jet

$$(6) \quad \omega^2 = ( T / \rho R_0^3 ) [ x I'_m (x) / I_m (x) ] ( x^2 + m^2 - 1 )$$

( where $\omega, T, \rho, R_0, x$ and $m$ have their usual meaning while $I_m (x)$ is the first kiind modified Bessel function of the order $m$ ) in the computer simulation to axisymmetric mode $m = 0$ and non-axisymmetric modes 1 and 2 .

The oscillation curves or the hollow cylinder ( for all $x$ values to $m=1$ and 2 and also to $m=0$ if $x \geqslant 1$ ) are little bit higher than those of the full jet in the same domains of stability .

The domain of instability is $0 < x < 1$ to $m=0$ only . The maximum temporal amplification prevailing the nonhollow case is for lower than that of the present model . The maximum modes cf instabillty ( $=0.34$ for full jet and $=0.82$ for hollow cylinder ) give fair feeling for that . This may be due to the fact that the pressure pushes in the same direction as the surface tension does. This reason is not oniy the cause of that since we have used volume conservation and the cylinder is filled with gas of ( constant ) strong equilibrium pressure . However, another reason can be stated if one takes into account the conservatin energy principle . The only source that can deliver energy is the potential energy due to surface tension . The greater instability is due to the fact that the same amount of potential energy can more easily give rise to motions, *i. e.*, increasing the kinetic energy according to the energy conservation principle (needing little kinetic energy ) because the hollow cylinder extends radially to infinity ; in comparison with the liquid cylinder ( which ) due to its radial iimitation needs comparatively large speeds in the z-direction in particular for long wavelengths $x << 1$ ) uses more kinetic energy for the same radial deformations .

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