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# SOME EXPANSION FORMULAE FOR THE MULTIVARIABLE H - FUNCTION

*By*

S. L. NAHTA
Department of Mathematics, Government College,
Barmer-344001, Rajasthan, India

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## ABSTRACT

The object of the present paper is to derive some expansion formulae for the multivariable $H$-function involving exponential functions, Legendre functions and Bessel functions .

## 1. INTRODUCTION

Srivastava and Panda [10] have defined the multivariable $H$-function as ( see also [9] and [11] )

$$H^{0,\,\varepsilon\,:\,(u',\,v')\,;\,\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,(B',\,D')\,;\,\ldots\,;\,(B^{(r)},\,D^{(r)})}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'],\ldots,[(b^{(r)}):\phi^{(r)}]\,;\\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{r)}]\,;\end{matrix}\; z_1,\ldots,z_r\right]$$

$$=\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r} U_1(s_1)\ldots U_r(s_r)\,V(s_1,\ldots,s_r)\, z_1^{s_1}\ldots z_r^{s_r}\,ds_1\ldots ds_r \qquad \ldots(1.1)$$

$$U_i(s_i) = \frac{\prod_{j=1}^{u^{(i)}} \Gamma(d_j^{(i)} - \delta_j^{(i)} s_i) \prod_{j=1}^{v^{(i)}} \Gamma(1 - b_j^{(i)} + \phi_j^{(i)} s_i)}{\prod_{j=1+u^{(i)}}^{D^{(i)}} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} s_i) \prod_{j=v^{(i)}+1}^{B^{(i)}} \Gamma(b_j^{(i)} - \phi_j^{(i)} s_i)}$$

$$i = 1, \ldots, r \qquad \ldots (1.2)$$

and

$$V(s_1, \ldots s_r) = \frac{\prod_{j=1}^{\epsilon} \Gamma(1 - a_j + \sum_{i=1}^{r} \theta_j^{(i)} s_i)}{\prod_{j=\epsilon+1}^{A} \Gamma(a_j - \sum_{i=1}^{r} \theta_j^{(i)} s_i) \prod_{j=1}^{C} \Gamma(1 - c_j + \sum_{i=1}^{r} \Psi_j^{(i)} s_i)} \qquad \ldots (1.3)$$

and, for convergence,

$$|\arg(z_i)| < \tfrac{1}{2} T_i \pi, \; i = 1, \ldots, r \qquad \ldots (1.4)$$

where,

$$T_i = -\sum_{j=\epsilon+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{v^{(i)}} \theta_j^{(i)} - \sum_{j=v^{(i)}+1}^{B^{(i)}} \theta_j^{(i)} - \sum_{j=1}^{C} \Psi_j^{(i)}$$

$$+ \sum_{j=1}^{u^{(i)}} \delta_j^{(i)} - \sum_{j=u^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0, \; 1 = 1, \ldots, r \qquad \ldots (1.5)$$

## 2. PRELIMINARY RESULTS

The following known results [ 7, p. 340 (95) ; 4, p. 316 ; 8, p. 78; 3, p. 62 ; 12, p. 324 ; 6, p. 291 (6) ] will be required in the proof of the integrals and expansions:

$$\int_{-\pi/2}^{\pi/2} (\cos\theta)^{m+n-2}\, e^{i(m-n)\theta}\, d\theta = \frac{\pi\Gamma(m+n-1)}{2^{m+n-2}\,\Gamma(m)\,\Gamma(n)}, \quad Re\,(m+n) > 1 \qquad \ldots (2.1)$$

$$\int_{-1}^{1} (1-x^2)^{\alpha-1}\, P_{\beta}^{\delta}(x)\,dx = \frac{2^{\delta}\pi\,\Gamma(\alpha+\delta/2)\,\Gamma(\alpha-\delta/2)}{\Gamma(\alpha+\frac{\beta}{2}+1)\,\Gamma(\alpha-\frac{\beta}{2})\,\Gamma(\frac{2-\delta+\beta}{2})\,\Gamma(\frac{1-\beta-\delta}{2})} \qquad \ldots (2.2)$$

$Re\,(2\alpha+\delta) > 0$, $Re\,(2\alpha-\delta) > 0$.

$$\int_{0}^{\infty} e^{ix}\, x^{\alpha}\, J_{\beta}(x)\, dx = \frac{e^{1/2(i(\alpha+\beta+1)\pi)}\,\Gamma(\alpha+\beta+1)\,\Gamma(-\alpha-\frac{1}{2})}{2^{\alpha+1}\,\Gamma(\frac{1}{2})\,\Gamma(\beta-\alpha)} \qquad \ldots (2.3)$$

$Re\,(\alpha+\beta) > -1$, $Re\,(\alpha) < -\frac{1}{2}$.

$$\int_{0}^{\infty} t^{s-1}\, J_{\beta}(\alpha t)\, J_{\delta}(\alpha t)\, dt = \frac{2^{s-1}\,\alpha^{-s}\, B(1-s, \frac{\beta+\delta+s}{2})}{\Gamma(\frac{\delta-\beta-s}{2}+1)\,\Gamma(\frac{\beta-\delta-s}{2}+1)} \qquad \ldots (2.4)$$

$-Re\,(\beta+\delta) < Re\,(s) < 1$, $\alpha > 0$.

$$\int_{a}^{b} e^{\frac{2m\pi ix}{a-b}}\, e^{\frac{2n\pi ix}{a-b}}\, dx = 0, \text{ if } m \neq -n \qquad \ldots (2.5)$$

$$= b-a, \text{ if } m = -n$$

$$\int_{-1}^{1} P_n^m(x)\, P_r^m(x)\, dx = 0, \text{ if } r \neq n$$

$$= \frac{2}{2n+1}\cdot\frac{(n+m)!}{(n-m)!} \quad \text{if } r = n \qquad \ldots (2.6)$$

$$\int_0^\infty t^{-1} J_{\alpha+2n+1}(t)\, J_{\alpha+2m+1}(t)\, dt = 0, \quad if\ m \neq n$$

$$= (4n+2a+2)^{-1}, \ if\ m = n \qquad \ldots (2.7)$$

$Re\,(a) + m + n > -1$.

## 3. MAIN INTEGRALS

$$\int_{-\pi/2}^{\pi/2} (\cos\theta)^{m+n-2}\, e^{i(m-n)\theta}\, H^{0,\,\varepsilon\,:\,(u',\,v')\,;\,\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,(B',\,D')\,;\,\ldots\,;\,(B^{(r)},\,D^{(r)})} \left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'];\,;\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \right.$$

$$\left. (e^{i\theta}\cos\theta)^{h_1/2}\, z_1, z_2, \ldots, z_r \right] d\theta$$

$$= \frac{2}{2^{m+n}\,\Gamma(n)} \sum_{i,-i} \frac{1}{i} H^{0,\varepsilon\,:\,(u'+1,\,v'+1)\,;\,(u'',\,v'')\,;\,\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A,C\,:\,(B'+2,\,D'+2)\,;\,(B'',\,D'')\,;\,\ldots\,;\,(B^{(r)},\,D^{(r)})} \left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'],(1,h_1), \\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'],(1,h_1), \end{matrix} \right.$$

$$\begin{matrix} (2-m-n,\frac{h_1}{2}) & ;[(b''):\phi''];\ldots;[(b^{(r)}):\delta^{(r)}]; \\ (1-m,\frac{h_1}{2}) & ;[(d''):\phi''];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix}$$

$$\left. \frac{e^{i\pi h_1}}{2h_1/2}\, z_1, z_2, \ldots, z_r \right] \qquad \ldots (3.1)$$

where $h_1$ is a positive number and $Re\,(m+n) > 1$ and conditions of (1.1) are also satisfied. The symbol $\sum_{i,-i}$ means that in the expression following it, $i$ is to be replaced by $-i$ and the two expressions so obtained must be added .

$$\int_{-1}^{1} (1-x^2)^{\alpha-1}\ P_{\beta}^{\delta}\ (x)\ H_{A,\,C\,:\,(B',\,D')\,;\,\ldots\,;\,(B^{(r)},\,D^{(r)})}^{0,\,\varepsilon\,:\,(u',\,v')\,;\,\ldots\,;\,(u^{(r)},\,v^{(r)})} \left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \right.$$

$$\left. z_1\,(1-x^2)^{h_1},\ \ldots,\ z_r\,(1-x^2)^{h_r} \right] dx$$

$$= \frac{2\delta\pi}{\Gamma(\frac{2-\delta+\beta}{2})\,\Gamma(\frac{1-\delta-\beta}{2})}\ H_{A+2,\,C+2\,:\,(B',\,D')\,;\,\ldots\,;\,(B^{(r)},\,D^{(r)})}^{0,\,\varepsilon+2\,:\,(u',\,v')\,;\,\ldots\,;\,(u^{(r)},\,v^{(r)})}$$

$$\left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}],\,(1-\alpha-\frac{\delta}{2};h_1,\ldots,h_r),\,(1-\alpha+\frac{\delta}{2};h_1,\ldots,h_r): \\ [(c):\Psi',\ldots,\Psi^{(r)}],\,(-\alpha-\frac{\beta}{2};h_1,\ldots,h_r),\,(1-\alpha+\frac{\beta}{2};h_1,\ldots,h_r): \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots,[(d^{(r)}):\delta^{(r)}]; \end{matrix}\ z_1,\ldots,z_r \right] \qquad \ldots(3.2)$$

where

$$|\arg(z_i)| < \tfrac{1}{2}\,T_i\,\pi,\ T_i > 0, \text{ and}$$

$$2\,Re\,(\alpha + \sum_{i=1}^{r} h_i\,\frac{d_j^{(i)}}{\delta_j^{(i)}}) > |\,\mathrm{Re}\,(\delta)\,|\,,\ j = 1,\ldots,u^{(i)}\,.$$

$$\int_0^\infty e^{ix} x^{\alpha} J_\beta(x)\, H^{0,\,\varepsilon\,:\,(u',\,v');\,\ldots;\,(u^{(r)},\,v^{(r)})}_{A,C\,:\,(B',\,D');\,\ldots;\,(B^{(r)},\,D^{(r)})}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; x^{h_1} z_1, z_2, \ldots, z_r\right] dx$$

$$\frac{e^{\frac{1}{2} i(\alpha+\beta+1)\pi}}{\Gamma(\frac{1}{2})\, 2^{\alpha+1}} H^{0,\,\varepsilon\,:\,(u'+1,\,v'+1);\,(u'',\,v'');\,\ldots;\,(u^{(r)},\,v^{(r)})}_{A,C\,:\,(B'+2,\,D'+1);\,(B'',\,D'');\,\ldots;\,(B^{(r)},\,D^{(r)})}$$

$$\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'],(-\alpha-\beta,h_1),(\beta-a,h_1);[(b''):\phi''];\\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'],(-\alpha-\tfrac{1}{2},h_1)\qquad;[(d''):\delta'']\end{matrix}\right.$$

$$\left.\begin{matrix}\ldots;[(b^{(r)}):\phi^{(r)}];\\ \ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\;\left(\frac{e^{\frac{1}{2}i\pi}}{2}\right)^{h_1} z_1, z_2, \ldots, z_r\right] \qquad \ldots (3.3)$$

where $|\arg(z_i)| < \frac{1}{2} T_i \pi,\ T_i > 0$, and

$$Re\left(\alpha+\beta+h_1 \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1,\ Re(\alpha) < \tfrac{1}{2},\ j=1,2,\ldots,u^{(i)}.$$

$$\int_0^\infty x^{-k} J_\delta(\alpha x) J_\beta(\alpha x)\, H^{0,\,\varepsilon\,:\,(u',\,v');\,\ldots;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,(B',\,D');\,\ldots;\,(B^{(r)},\,D^{(r)})}$$

$$\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'];\ldots;(d^{(r)});\delta^{(r)}];\end{matrix}\right.$$

$$\left. z_1 x^{2h_1}, z_2, \ldots, z_r\right] dx$$

$$= \frac{1}{2\sqrt{\pi}} \alpha^{k-1} H^{0,\varepsilon : (u'+2,\, v'+1);\, (u'',\, v'');\, \ldots;\, (u^{(r)},\, v^{(r)})}_{A,\, C : (B'+4,\, D'+2);\, (B'',\, D'');\, \ldots;\, (B^{(r)},\, D^{(r)})} \left[ \begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'], \\ [(c) : \Psi', \ldots, \Psi^{(r)}] : [(d') : \delta'], \end{matrix} \right.$$

$$\left(\frac{\beta}{2} - \frac{\delta}{2} + \frac{k}{2} + \frac{1}{2},\, h_1\right),$$

$$\left(\frac{k}{2},\, h_1\right),\, \left(\frac{k}{2} + \frac{1}{2},\, h_1\right)$$

$$\left(\frac{1}{2} + \frac{k}{2} - \frac{\beta}{2} - \frac{\delta}{2},\, h_1\right),\, \left(\frac{1}{2} + \frac{\delta}{2} + \frac{\beta}{2} + \frac{k}{2},\, h_1\right),$$

$$\left(\frac{\delta}{2} - \frac{\beta}{2} + \frac{k}{2} + \frac{1}{2},\, h_1\right);\, [(b'');\, \phi''];\, \ldots;\, [(b^{(r)}) : \phi^{(r)}];$$

$$;\, [(d'') : \phi''];\, \ldots;\, [(d^{(r)}) : \delta^{(r)}];$$

$$\left. z_1 \alpha^{-2h_1},\, z_2,\, \ldots,\, z_r \right] \qquad \ldots(3.4)$$

where $|\arg(z_i)| < \frac{1}{2} T_i \pi$, $T_i > 0$, and

$$Re\left(\beta + \delta - k + 2h_1 \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1,$$

$$Re\left(2h_1 \frac{d_j^{(i)}}{\delta_j^{(i)}} - k\right) < 2h_1,\ j=1 \ldots,\ u^{(i)}$$

**Proof :** Equation (3.1) can be proved on using the contour integral (1.1), interchanging the order of integration, which is justified under the conditions stated with the result, evaluating the snner integral with the help of (2.1) and using a relation $\Gamma(\alpha)\,\Gamma(1-\alpha) = \frac{\pi}{sin\ \pi\alpha}$

$= \frac{2\pi i}{e^{i\pi\alpha} - e^{-i\pi\alpha}}$ , and applying the definition (1 1) .

The proof of the formula (3.2) to (3.4) can be developed by proceeding on similar lines with the help of the result (2.2) to (2.4) respectively in place of result (2.1) and a relation $\Gamma(\alpha)\,\Gamma(1-\alpha) = \frac{\pi}{sin\ \pi\alpha}$ .

## 4. EXPANSION FORMULAE

$$(\cos\theta)^{2n-2}\, H^{0,\ \varepsilon\ :\ (u',\ v')\ ;\ \ldots\ ;\ (u^{(r)},\ v^{(r)})}_{A,C\ :\ (B',\ D')\ ;\ \ldots\ ;\ (B^{(r)},\ D^{(r)})} \left[\begin{matrix} [\,(a) : \theta', \ldots, \theta^{(r)}\,] : [\,(b') : \phi'\,] ; \ldots ; [\,(b^{(r)}) : \phi^{(r)}\,] ; \\ [\,(c) : \Psi', \ldots, \Psi^{(r)}\,] : [\,(d') : \delta'\,] ; \ldots ; [\,(d^{(r)}) : \delta^{(r)}\,] ; \end{matrix} (e^{i\theta} \cos\theta)^{h_1/2}\ z_1, z_2, \ldots, z_r \right]$$

$$= 2\left\{ \sum_{R=-\infty}^{\infty} \frac{1}{2^{2R}\Gamma(R)} \sum_{i,-i} \frac{1}{i} \right.$$

$$H^{0,\ \varepsilon\ :\ (u'+1,\ v'+2)\ ;\ (u'',\ v'')\ ;\ \ldots\ ;\ (u^{(r)},\ v^{(r)})}_{A,C\ :\ (B'+2,\ D'+2)\ ;\ (B'',\ D'')\ ;\ \ldots\ ;\ (B^{(r)},\ D^{(r)})}$$

$$\left[\begin{matrix} [\,(a) : \theta', \ldots, \theta^{(r)}\,] : [\,(b') : \phi'\,], (1, h_1), (2-2R, \frac{h_1}{2}) ; [\,(b'') : \phi''\,] ; \\ [\,(c) : \Psi', \ldots, \Psi^{(r)}\,] : [\,(d') : \delta'\,], (1, h_1), (1-R, \frac{h_1}{2}) ; [\,(d'') : \delta''\,] ; \end{matrix}\right.$$

$$\left.\left.\begin{matrix} \ldots ; [\,(b^{(r)}) : \phi^{(r)}\,] ; \\ \ldots ; [\,(d^{(r)}) : \delta^{(r)}\,] ; \end{matrix} \frac{e^{i\pi h_1}}{2^{h_1/2}}\ z_1, z_2, \ldots, z_r \right]\right\} e^{-2i(R-n)\theta} \qquad \ldots (4.1)$$

where $h_1$ is a positive number, $-\pi/2 < \theta < \pi/2$, and $Re\,(n) > 1/2$.

$$(1-x^2)^{\alpha-1}\; H^{0,\,\varepsilon\,:\,(u',\,v')\,;\,\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,(B',\,D')\,;\,\ldots\,;\,(B^{(r)},\,D^{(r)})} \left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \; z_1(1-x^2)^{h_1},\ldots,z_r(1-x^2)^{h_r} \right]$$

$$= 2^{\delta-1}\pi \sum_{R=0}^{\infty} \frac{(2R+1)(R-\delta)\,!}{(R+\delta)\,!\;\Gamma\left(\frac{2-\delta+R}{2}\right)\Gamma\left(\frac{1-\delta-R}{2}\right)}$$

$$H^{0,\,\varepsilon+2\,:\,(u',\,v')\,;\,\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A+2,\,C+2\,:\,(B',\,D')\,;\,\ldots\,;\,(B^{(r)}\,D^{(r)})}$$

$$\left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}],(1-\alpha-\frac{\delta}{2};h_1,\ldots,h_r),(-\alpha+\frac{\delta}{2};h_1,\ldots,h_r); \\ [(c):\Psi',\ldots,\Psi^{(r)}],(-\alpha-\frac{R}{2};h_1,\ldots,h_r),(1-\alpha+\frac{R}{2};h_1,\ldots,h_r); \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots,[(d^{(r)}):\delta^{(r)}]; \end{matrix} \; z_1,\ldots,z_r \right] P_R^{\delta}(x) \qquad \ldots(4.2)$$

where

$$|\,arg\,(z_i)\,| < \tfrac{1}{2}\,T_i\,\pi,\; T_i > 0, \text{ and}$$

$$Re\left(a + \sum_{i=1}^{r} h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > |\,\mathrm{Re}\,(\delta)\,|,\; j = 1,\ldots,u^{(i)};$$

$$x^{\alpha}\; H^{0,\,\varepsilon : (u',\,v');\,\ldots;\,(u^{(r)},\,v^{(r)})}_{A,\,C : (B',\,D');\,\ldots;\,(B^{(r)},\,D^{(r)})}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}] \\ [(c):\Psi',\ldots,\Psi^{(r)}]\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\quad x^{h_1} z_1,\, z_2,\,\ldots,\, z_r\right]$$

$$= \frac{1}{\Gamma(\frac{1}{2})2^{\alpha-1}} \sum_{R=0}^{\infty} e^{\frac{1}{2}i(\alpha+\eta)\pi}$$

$$\eta\, H^{0,\,\varepsilon : (u'+1,\,v'+1);\,(u'',\,v'');\,\ldots;\,(u^{(r)},\,v^{(r)})}_{A,C : B'+2,\,D'+1);\,(B'',\,D'');\,\ldots;\,(B^{(r)},\,D^{(r)})}$$

$$\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'],(-\alpha+1-\eta\, h_1), \\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'],(-\alpha+\frac{1}{2},h_1)\end{matrix}\right.$$

$$\begin{matrix}(\eta-\alpha+1\,;h_1);[(b''):\phi''];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ ;[(d''):\delta''];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}$$

$$\left.\left(\frac{e^{\frac{1}{2}i\pi}}{2}\right)^{h_1} z_1,\, z_2,\,\ldots,\, z_r\right] J_{\eta}(x) \qquad \ldots (4.3)$$

where $\eta = n + 2R + 1$,

$|\arg(z_i)| < \frac{1}{2} T_i\, \pi,\ T_i > 0$, and

$Re\,(\alpha + h_1 \frac{d_j^{(i)}}{\delta_j^{(i)}}) > 0,\ Re\,\alpha < \frac{1}{2},\ j=1,\ldots,u^{(i)}.$

$$x^{-k} J_{\beta}(ax)\; H^{0,\,\varepsilon : (u',\,v');\,\ldots;\,(u^{(r)},\,v^{(r)})}_{A,C : (B',\,D');\,\ldots;\,(B^{(r)},\,D^{(r)})}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\Psi',\ldots,\Psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\quad z_1\, x^{2h_1},\, z_2,\,\ldots,\, z_r\right]$$

$$= \frac{\alpha^k}{\sqrt{\pi}} \sum_{R=0}^{\infty} \eta \, H^{0,\,\varepsilon\,:\,(u'+2,\,v'+1)\,;\,(u'',\,v'')\,;\,\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A,\,C\,:\,(B'+4,\,D'+2)\,;\,(B'',\,D'')\,;\,\ldots\,;\,(B^{(r)},\,D^{(r)})}$$

$$\left[\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'],\left(\frac{\beta}{2}-\frac{\eta}{2}+\frac{k}{2}+1,h_1\right), \\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'],\left(\frac{k}{2}+\frac{1}{2},h_1\right) \end{matrix}\right. ,$$

$$\left(1+\frac{k}{2}-\frac{\beta}{2}-\frac{\eta}{2},h_1\right),\left(1+\frac{\eta}{2}+\frac{\beta}{2}+\frac{k}{2},h_1\right),\left(\frac{\eta}{2}-\frac{\beta}{2}+\frac{k}{2}+1,h_1\right);$$

$$\left(\frac{k}{2}+1,\ h_1\right) \qquad ;$$

$$\left.\begin{matrix} [(b''):\phi''];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d''):\delta''];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \; z_1\alpha^{-2h_1}, z_2, \ldots, z_r \right] J_n(x), \quad \ldots (4.4)$$

where $\eta = u + 2R + 1$, $|\, arg\,(z_i)\,| < \frac{1}{2} T_i \pi$, $T_i > 0$, and

$$Re\left(\beta - k + 2h_1 \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0,$$

$$Re\left(2h_1 \frac{b_j^{(i)}}{\phi_j^{(i)}} - k\right) < 2h_1 + 1,\ j=1,\ldots,u^{(i)}.$$

**Proof :** To prove (4.1), let

$$f(\theta) = (\cos\theta)^{2n-2}\, H^{0,\,\varepsilon\,:\,(u',\,v')\,;\,\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A,C\,:\,(B',\,D')\,;\,\ldots\,;\,(B^{(r)},\,D^{(r)})}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\Psi',\ldots,\Psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)});\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)});\delta^{(r)}];\end{matrix}\; (e^{i\theta}\cos\theta)^{h_1/2} \;\; z_1, z_2, \ldots, z_r\right]$$

$$= \sum_{R=-\infty}^{\infty} C_R \, e^{-2i(R-n)\theta}, \qquad \ldots (4.5)$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad Re(n) > 0 .$$

Equation (4.5) is valid, since $f(\theta)$ is continuous and of bounded variation in the open intervals $(-\frac{\pi}{2} \text{ to } \frac{\pi}{2})$.

Now integrating (4.5) with respect to $\theta$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we get

$$\int_{-\pi/2}^{\pi/2} f(\theta)\, d\theta = \sum_{R=-\infty}^{\infty} C_R \int_{-\pi/2}^{\pi/2} e^{-2i(R-n)\theta} \, d\theta$$

Now using (3.1) and the orthogonality property of exponential functions (2.5), we obtain

$$C_n = \frac{2}{2^{2n}\,\Gamma(n)} \sum_{i,-i} \frac{1}{i}$$

$$H^{0,\,\varepsilon\,:\,(u'+1,\; v'+2)\,;\,(u'',\; v'')\,;\; \ldots\,;\,(u^{(r)},\; v^{(r)})}_{A,C\,:\,B'+2,\; D'+2)\,;\,(B'',\; D'')\,;\;\ldots\,;\,(B^{(r)},\; D^{(r)})}$$

$$\left[\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'], (1, h_1), (2-2n, h_1/2) ; \\ [(c) : \Psi', \ldots, \Psi^{(r)}] : [(d') : \delta'], (1, h_1), (1-n, h_1/2) ; \end{matrix}\right.$$

$$[(b'') : \phi''] ; \ldots ; [(b^{(r)}) : \phi^{(r)}] ;$$

$$[(d'') : \delta''] ; \ldots ; [(d^{(r)}) : \delta^{(r)}] ;$$

$$\frac{e^{i\pi h_1}}{2^{h_1/2}}) \quad z_1, z_2, \ldots, z_r \,] \qquad \ldots (4.6)$$

With the help of equations (4-5) and (4.6), the formula (4.1) follows. The proofs of the formulas (4.2) to (4.4) can be developed by proceeding on similar lines with the help of (3.2) and (2.6), (3.3) and (2.7), (3.4) and (2.7), respectively.

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