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FIXED POINTS FOR MULTI - VALUED MAPPINGS IN LOCALLY CONVEX SPACE

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ABSTRACT

Let Q be a family of continuous seminorms determining the topology of a locally convex topological vector space E. In this paper, sufficient conditions are given for the existence of fixed points for multivalued mappings. The results of this paper strengthen those of Kirk, Assad and Kirk Markin, and others. Our results are motivated by the results of Samanta (12) and extend his results to locally convex spaces.

1. INTRODUCTION

Let X be a metric space and (2^X. H) be the space of all nonempty compact subsets of X endowed with the Hausdorff metric H. Let $T : X \to 2^X$ be a mvltivalued mapping. A point $x \epsilon X$ is said to be a fixed point of T, provided $x \in T(x)$. Kirk [3] proved the following ;

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If K is a nonempty, closed, bounded, and convex subset of a reflexive Banach space X and if K possesses normal structure, then every self-nonexpansive mapping of K has a fixed point, Markin [8] extended the result of Kirk to multivalued mappings. Markin's result is : Let X be a real Hilbert space and $T : X \to 2^X$ be nonexpansive. If

$T(x) \subset B$ for $x \in B$, where $B = \{ x \in X ; \|x\| \leqslant 1 \}$,

then T has a fixed point .

Assad and Kirk [1] have extended this result of Markin to Banach spaces satisfying Opial's condition. Fixed points for set-valued non-expansive mappings and for set-valued Kannan-type mappings in locally convcx spaces were given by Su and Sehgal [11] and Ko and Tsai [5], respectively. Our results, which are primarily motivated by the work of Ko [6], Ko and Tsai [5], and Su aud Sehgal [11], extend the results of Assad and Kirk [1], Kirk [4], Markin [8], Nadler [9], and others .

2. PRELIMINARIES AND NOTATIONS

Throguhout this paper, E will denote a Hausdorff, locally convex, linear topological space and Q, a (fixed) family of continuous semi-norms whic hgenerate the topology of E. For any $p \in Q$ and $A \subseteq E$, $\delta_p(A)$ will denote the p-diameter of A, that is, $\delta_p(A) = sup \{ p(x-y) : x, y \in A \}$, $B_p(x, r)$ ($B_p[x, r]$) will denote the open (closed) ball with center at X and radius r. For $A \subseteq E$, $d_p(x, A) = inf \{ p(x-y) : y \in A \}$. We shall denote by 2^E, the space of all nonempty compact subsets of E.

Definition 2. 1 A convex subset K of E is said to have *normal structure* if for every closed, bounded, convex subset W of K such that $\delta_p(W) > 0$, there exists $x \in W$ such that $sup \{ p(x-y) : y \in W \} < \delta_p(W)$.

Definition 2.2 Let K be a nonempty subset of E, and let $\{ W_\alpha : \alpha \in \triangle \}$ be a decreasing set of nonempty bounded subsets of K. For $x \in K$, $\alpha \in \triangle$, and $p \in Q$, define

$$r_{p,\alpha}(x) = \sup \{ p(x - y) : y \in W_\alpha \}$$
$$r_p(x) = \inf \{ r_{p,\beta}(x) : \beta \in \triangle \}$$
$$r_p = \inf \{ r_p(x) : x \in K \}$$
$$M = \{ x \in K : r_p(x) = r_p \} .$$

The set M and the number r_p will be called the asymptotic center and the asymptotic radii, respectively, of $\{ W_\alpha : \alpha \in \triangle \}$ in K with respect to p .

Definition 2.3 Let K be a nonempty subset of E. $T : K \to 2^K$ is said to be a *nonexpansive map* if for each $p \in Q$, for each $x, y \in K$, and for each $x_1 \in T(x)$, there exists $y_1 \in T(y)$ such that $p(x_1 - y_1) \leqslant p(x - y)$.

Note that if the set K is compact, then each set A in 2^K is compact and, hence, bounded under each $p \in Q$.

Thus, the Hausdorff metric D_p [2] induced from the seminorm p is well defined on 2^K . It should be noted that D_p is a metric on 2^K, although p is a seminorm on K. Hence, if K is compact, Dcfinition 2.3 is equivalent to the following :

Definition 2.4 Let K be a compact subset of E. A map $T : K \to 2^K$ is said to be a nonexpansive map if for each $p \in Q$ and $x, y \in K$, we have $D_p(Tx, Ty) \leqslant p(x - y)$.

3. RESULTS

Theorem 3.1 Let E be a semireflexive, locally convex Hausdorff topological vector space and K be a closed, bounded, convex subset of E, possessing normal structure . If $T : K \to 2^E$ is a mapping such that

(i) $T(x) \cap K = \phi$ for all $x \in K$,

(ii) for any closed convex subset L of K satisfying

$T(w) \cap L = \phi$ for all $w \in L$,

$D_p(T(x) \cap L, T(y) \cap L) \leqslant p(x-y)$ whenever

$x, y \in L$, $x \neq y$. Then T has a fixed point.

Proof Let G be the family of all nonempty, closed convex subsets M of K such that $T(x) \cap M \neq \phi$ for all $x \in M$. Then, G is nonempty, since $K \in G$. Partially order G by inclusion. Let $F = \{F_i\}_{i \in \triangle}$ be a decreasing chain in G. It follows that $\bigcap_{i \in \triangle} F_i \neq \phi$ (due to the semireflexivity of E). Let $F_0 = \bigcap_{i \in \triangle} F_i$, a nonempty, closed convex subset of M. Let $x \in F_0$. By hypothesis, $T(x) \cap (\bigcap_{i \in \Delta} F_i) \neq \phi$. For any $F_i \in F$, $T(x) \cap F_i$ is compact and $\{T(x) \cap F_j\}_{j \geqslant i}$ is a family of nonempty closed subsets of the compact set $T(x) \cap F_i$ having finite intersection property. Consequently, $\bigcap_{j \geqslant i} (T(x) \subset F_j) \neq \phi$ and, therefore, $\bigcap_{i \in \triangle} (T(x) \cap F_i) \neq \phi$, *i. e.*, $T(x) \cap F_0 \neq \phi$. Thus, any chain in G has a greatest lowerbound, and by Zorn's Lemma, there is a minimal member N in G.

We claim that N is a singleton set. If not, then as shown by Lim [7, p 315–316], the eenter of N, denoted by R is a nonempty proper closed convex subset of N. It is enough to show that R belongs to G. Let $y \in R$, then $y \in N$ and $T(y) \cap N \neq \phi$. Take $z \in T(y) \cap N$. Let $W = B_p[z, r(N)] \cap N$. Then, W is a nonempty closed convex subset of N. Let $x \in W$. Using Hypothesis (ii), we have $D_p(T(x) \cap N, T(y) \cap N) \leqslant p(x-y)$.

Since $z \in T(y) \cap N$, there is a $u \in T(y) \cap N$ such that

$p(u-y) \leqslant D_p(T(x) \cap N, T(y) \cap N) \leqslant p(x-y)$. Hence,

$T(x) \cap W \neq \phi$ for all $x \in W$. By the minimality of N, we have

$N \subseteq W$, that is, $N \subseteq B_p[z, r(N)]$. Consequently, $z \in R$ and

we have $T(y) \cap R \neq \phi$ for all $y \in R$. Therefore $R \in G$, a contradiction to the minimality of N. Hence, N consists of a single point which is a fixed point of T.

The following example shows that Condition (ii) cannot be replaced by $D_p(Tx, Ty) \leqslant p(x-y)$ for all $x, y \in K$, $x \neq y$.

Example 2.1 Let $X = R^2$ with usual norm and $K = \{(x,0) : 0 \leqslant x \leqslant 1\}$. Then, K is a bounded closed convex subset of a reflexive Banach space X. Define $T : K \to 2^X$ as follows :

$$T(x,0) = \{(\frac{1}{\sqrt{2}}(\tfrac{1}{2} - x), \frac{1}{\sqrt{2}}(\tfrac{1}{2} - x)), (x + \tfrac{1}{2}, 0)\} \quad 0 \leqslant x \leqslant \tfrac{1}{2},$$

$$T(x + \tfrac{1}{2}, 0) = \{(1 - \frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}), (x, 0)\}, 0 \leqslant x \leqslant \tfrac{1}{2},$$

Then $T(x)$ intersects K for every x in K. If

$(x, 0), (y, 0) \in K$ such that $x, y \in [0, \tfrac{1}{2}]$ or $x, y \in [\tfrac{1}{2}, 1]$,

then $D(T(x, 0), T(y, 0)) \leqslant \|(x, 0) - (y, 0)\|$. If

$x, y \in [0, \tfrac{1}{2}]$, then $D(T(x, 0), T(y + \tfrac{1}{2}, 0))$

$$\leqslant [(\frac{1}{\sqrt{2}}(\tfrac{1}{2} - x) - y)^2 + (\frac{1}{\sqrt{2}}(\tfrac{1}{2} - x)^2]^{1/2}$$

$$= [x^2 + y^2 + \sqrt{2}xy - (x + \frac{y}{\sqrt{2}} + \frac{1}{4}]^{1/2},$$

whenever $(\sqrt{2}+2)\,xy \leqslant (1+\frac{1}{\sqrt{2}})\,y$, *i.e.*, $x \leqslant \frac{1}{2}$ with $y \neq 0$.

Thus, $D(T(x, 0), T(y, 0)) \leqslant \|(x, 0) - (y, 0)\|$ for all $(x,0)$, $(y, 0)$, $x \neq y$ in K. Now for $x,y \in [0, \frac{1}{2}]$,

$$D(T(x, 0) \cap K, T(y+\tfrac{1}{2}, 0) \cap K) = \|(x+\tfrac{1}{2}, 0) - (y, 0)\|$$
$$= \|x+\tfrac{1}{2}-y\|$$
$$= \|y+\tfrac{1}{2}-x\|$$

whenever $x > y$. So T is nonexpansive and satisfies all but Condition (ii) of Theorem 2.1. Clearly, T has no fixed point.

For the proof of Theorem 2.2, we need to recall the following :

Lemma 2.1 [10, Lemma 3.1]. Let H and K be two closed subsets of E such that $H \cap K \neq \phi$. If H is convex, then $\partial_H(K) = \phi$ if and only if $H \subseteq K$ (here, $\partial_H(K)$ denotes the relative boundary of $H \cap K$ in H).

Theorem 2.2 Let E be a Housdorff, locally convex topological vector space and K be a nonempty, closed convex subset of E Let L be a nonempty, weakly compact, convex subset of K possessing normal structure and $T : L \to 2^E$ be a mapping satisfying $T(x) \cap K \neq \phi$ for all $x \in L$. Further, for M a nonempty, closed convex subset of K such that $T(x) \cap M \neq \phi$, whenever $X \in M \cap L$, the following conditions hold :

(i) $D_p(T(x) \cap M, T(y) \cap M) \leqslant p(x-y)$ for all $x, y \in M \cap L$, for all $p \in Q$, and

(ii) $T(W) \cap (M \cap L) \neq \phi$, where $w \in \partial_M(L)$.

Then, T has a fixed point.

Proof Let G be the family of all nonempty closed convex subsets M of K such that $M \cap L \neq \phi$ and $T(x) \cap M \neq \phi$ whenever $x \in M \cap L$. Then G is nonempty, since $K \in G$. Partially order G by inclusion. Arguing similarly, as in the proof of Theorem 2.1, it follows that every chain in G has a greatest lower bound. So, by Zorn's Lemma, there is a minimal element N in G. If $\partial_N (L) = \phi$, then, by Lemma 2.1, $N \subseteq L$. In this case, Theorem 2.1 is applicable, and T has a fixed point. Thus, we assume that $\partial_N (L) \neq \phi$ and show that $N \cap L$ is a singleton set. Let $\delta_P(N \cap L) > 0$, for some p.

Since L has normal structure, there is a point $y \in L \cap N$ such that

$$0 < \alpha = \sup_{x \in N \cap L} p(x - y) < \delta(N \cap L). \qquad (1)$$

Consider the set $H = \{ x \in N : N \cap L \subseteq B_P [x, \alpha] \}$, a closed convex subset of N. Since $y \in H$, H is nonempty and, by (1) $H \neq N$. We show that $H \in G$. Let $z \in H \cap L$ and

$y \in \partial_N(L)$. Then, $y \in N \cap L$ and, hence,

$p(z - y) \leqslant \alpha$. Now, since $N \in G$ from Hypothesis (i), we have for $y, z \in N \cap L$,

$$D_p (T(y) \cap N, T(z) \cap N) \leqslant p(y - z). \qquad (2)$$

Consequently, $D_P (T(y) \cap N, T(z) \cap N) \leqslant \alpha$. Since $y \in \partial_N(L)$, by Hypothesis (ii), $T(y) \cap (N \cap L) \neq \phi$.

Take $u \in T(y) \cap (N \cap L)$. Then, using (2), we obtain for $v \in T(z) \cap N, p(u-v) \leqslant D_P (T(z) \cap N, T(y) \cap N) \leqslant p(z-y)$.

Let $W = B_P [v, \alpha] \cap N$. Then, W is a closeed convex subset of K with $W \cap L \neq \phi$ (since $u \in W \cap L$). If $w \in W \cap L \cap N \cap L$, then $p(z-w) \leqslant \alpha$.

Further, $D_p(T(W) \cap N, T(z) \cap N) \leqslant p(z-w) \leqslant \alpha$.
Since $v \in T(z) \cap N$, there exists an element $s \in T(w) \cap N$ such that
$p(v-s) \leqslant D_p(T(w) \cap N, T(z) \cap N) \leqslant p(w-z) \leqslant \alpha$.
This shows that $s \in W$, or that $T(w) \cap W \neq \phi$ for all
$w \in W \cap L$. Therefore, $W \in G$. By the minimality of N, we have
$N = W$. Hence, $N \cap L \subseteq B_P[v, \alpha]$ and $v \in H$.
We now have $T(z) \cap H \neq \phi$ for all $z \in H \cap L$, and it follows that
$H \in G$. But H is a proper subset of N, a contradiction. Hence, $N \cap L$ is a singleton set. Let $N \cap L = \{x\}$.
By assumption, $\phi \neq \partial_N (L \subseteq N \cap L$, $\partial_N(L) = \{x\}$.

Now, by Hypothesis (ii), $x \in T(x)$. This con pl tes the prcof.

Remark 2.1 The example below shows that Condition (ii) of Theorem 2.2 cannot be replaced by the following coudition :
(ii)′ If $x \in \partial_K(T)$, then $T(x) \cap T \neq \phi$.

Example 2.2 Let $X = R^2$ with the usual norm and
$K = \{(x, 0) : 0 \leqslant x < \infty\}$ and
$L = \{x, 0) : 0 \leqslant x \leqslant 1\}$. Define $T : L \to 2^X$
by $T(x,0) = \{(1 + x, 0), (0, 1 - x)\}$, $(x\ 0) \in L$.
Tben K is a closed and convex subset of a reflexive Banach space, and L is a closed, bounded convex subset of K. The mapping T satisfies the condition $T(x) \cap K \neq \phi$ for all $x \in L$, Condition (i) of Theorem 2.2 and Condition (ii)′ are also satisfied. By taking $M = [1/2, \infty)$, we see that Hypothesis (ii) of Theorem 2.2 is not satisfied. Clearly, T does not have a fixed point.

Remark 2.2 Our Theorem 2.2 is applicable to $L_p[0,1]$, $1 < p < 2$. However, the results of Markin, and Assad and Kirk, do not hold there, since $L_p[0, 1]$ is not a Hilbert space, nor does it satisfy Opial's condition.

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