Jñānabha, Vol. 18, 1988

FIXED AND COMMON FIXED POINTS IN CONVEX METRIC SPACES

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(*Received* : *March 8, 1988*)

Recently many authors have proved fixed point theorems)see for example [1], [3], [5], [9]) for operators mapping a Banach space X into itself. In each of these theorems the operators involved are nonexpansive *i. e.* if T maps the Banach space X into itself, then

(a) $\| Tx-Ty \| \leqslant \| x-y \|$, for $x, y \in X$

Kannan [7], [8] proved some fixed point theorems for operators mapping a Banach space into itself which, instead of beiag nonexpansive possess the following : if T is a mapping of a Banach space X into itself, then

(b) $\| Tx-Ty \| \leqslant 1/2 \; [\| x-Tx \| + \| y-Ty \|]$ for $x, y \in X$.

It may be observed that condition (a) implies the continuity of the operaters in the whole space while condition (b) has no such implication .

The main aim of the present paper is to generalize the results of Kannan [7], [8] to convex metric spaces. We also extend the results of Kirk [9], Goebel, Kirk and Shimi [6] to the setting of convex metric spaces.

In the sequel we use I for the iaterval [0, 1]. The following notion was introduced by Takahashi [11].

Definition 1.1. Let X be a metric space. A mapping $W : X \times X \times I \to X$ is said to be a *convex structure* on X if for all $x, y \in X$ and $\lambda \in I$ the following condition is satisfied :

$$d(u, W(x, y; \lambda)) \leqslant \lambda d(u, x) + (1-\lambda) d(u, y)$$

for all u in X. We will call a metric space with a convex structure a *convex metric* space.

If X is a Banach space, then, as a metric space with $d(x, y) = \|x-y\|$, the mapping $W : X \times X \times I \to X$ defined by $W(x, y; \lambda) = \lambda x + (1-\lambda) y$ is a convex structure,

Definition 1. 2. Let X be a convex metric space. Let K be a nonempty subset of X. we say that K is *convex* if $W(x, y; \lambda)$ belongs to K for all x, y in K and $\lambda \in I$.

Definition 1. 3. Let X be a metric space and E be a nonempty subset of X, then we define

$$R_x(E) = sup\{ d(x,y) : y \in E \}$$

$$R(E) = inf\{ R_x(E) : x \in E \}$$

$$E_C = \{ x \in E : R_x(E) = R(E) \}.$$

Definition 1. 4. Let X be a convex metric space. We say that X has *property* (C) if every bounded decreasing net of nonempty closed and convex subsets of X has a nonempty intersection.

It follows from Smulian's theorem that every weakly compact convex subset of a Banach space has property (C) [4, p. 433] .

Lemma 2.1. Let X be a convex metric space. If X has property (C), then E_C is nonempty, closed and convex .

Proof. For each $n \in N$, let $C(x,n) = \{ y \in E : d(x, y) \leqslant R(E)+\frac{1}{n}\}$ and $F_n = \bigcap_{x \in E} C(x, n)$. It follows that $\{F_n\}$ is a decreasing sequence of nonempty, closed and convex sets. Since X has property (C), $E_C = \bigcap_{n=1}^{\infty} F_n$ is nonempty, closed and convex .

Definition 2. 2. Let X be a metric space. A mapping $T : X \to X$ is said to satisfy *condition* (A) if for all $x, y \in X$ we have

$$d(Tx, Ty) \leqslant max\{ d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/3, [d(x, y) + d(x, Tx) + d(y, Ty)]/3\},$$

The proof of the next theorem is similar to that used by Kirk [9] in Banach space . The theorem extends Kirk's theorem and Theorem 1 of Kannan [7] .

Theorem 2. 3. Let X be a convex metric space having property (C). Let D be a nonempty, bounded, closed and convex subset of X. Let $T : D \to D$ be a mapping staisfying condition (A) with the property

$$\sup_{z \in K} d(z, Tz) \leqslant \frac{\delta(K)}{2},$$

where K is any nonempty convex subset of D which is mapped into itself by T. Then T has a unique fixed point in D .

Proof. Let H be the family of all nonempty, closed and convex subsets of D which is mapped into itself by T. By property (C) and Zorn's Lemma it follows that H has a minimal element, say E.

By Lemma 2.1, E_C is nonempty, closed and convex For $x \in E_C$ and $y \in E$ we have

$$d(Tx, Ty) \leqslant max \{ d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/3,$$

$$[d(x,y) + d(x, Tx) + d(y, Ty)] / 3 \}$$

$$\leqslant max \{ \frac{\delta(E)}{2}, [d(x, y) + d(y, Ty) + d(y\ x) + d(x, Tx)] / 3,$$

$$[d(x, y) + \delta(E)] / 3$$

$$\leqslant max \{ \frac{\delta(E)}{2}, [2d(x, y) + \delta(E)]/3, [d(x, y) + \delta(E)]/3 \}$$

$$\leqslant sup \{ d(x, y) : y \in E \} = R_x(E) = R(E).$$

So, $T(E)$ is contained in a closed spherical ball

$S(T\lambda, R(E))$ and hence $T(E \cap \bar{S}) \subseteq E \subseteq \bar{S}$ (as $T(E) \subseteq E$). By the minimality of E, we conclude that $E \subseteq \bar{S}$.

Hence,

$$d(Tx, y) \leqslant R(E)$$

for all y in E. So,

(1) $sup \{ d(Tx, y) : y \in E \} \leqslant R(E)$.

Hence,

$R_{T(x)}(E) = sup \{ d(Tx, y) : y \in E \} \leqslant R(E)$,
which implies that

(2) $R_{T(x)}(E) = R(E)$, *i. e.* $T(x)$ belongs to E_C. Therefore, we have proved that $T(E_C) \subseteq E_C$.

We assert that, *if* E contains more than one element, then E_C is a proper subset of E. Suppose not, *i. e.* $E = E_C$. Then for x, y in E,

$$R_x(E) = R_y(E) = R(E).$$

Therefore, $sup\{d(x, z) : z \in E\} = sup\{d(z, y) : z \in E\}$

for all x, y iu E. From this it follows that $sup\{d(x,z) : z \in E\} = R(E)$, a constant for all $x \in E$, Hence $\delta(E) = sup\{d(x,z) : x, z \in E\} = R(E)$.

This implies that for $x \in E = E_C$ (and x in E_C)

(3) $sup\{d(Tx, z) : z \in E\} = \delta(E)$.

For x, y in $E = E_C$, by the definition of T we have

$$d(Tx, Ty) \leqslant max\{d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/3, [d(x,y) + d(x, Tx) + d(y, Ty)]/3\}$$

$$\leqslant max\{\frac{\delta(E)}{2}, 2/3\,\delta(E), [\delta(E) + \delta(E)]/3\}$$

$$\leqslant 2/3\,\delta(E).$$

Therefore $T(E_C)$ is contained in a closed spherical ball $S(Tx, 2/3\,\delta(E))$ and hence $T(E_C \cap \bar{S}) \subseteq E_C \cap \bar{S}$ (as $T(E_C) \subseteq (E_C)$). Then by the minimality of $E, E = E_C$, we comclude that $E_C \subseteq \bar{S}$. Hence $d(Tx, y) \leqslant 2/3\,\delta(E)$ for all y in E. So, $sup\{d(Tx, y) : y \in E\} \leqslant 2/3\,\delta(E)$, which contradicts (3), because E contains more than one element. Thus, we conclude that if E contains more than one element, then E_C is a proper subset of E. But then in view of (2) it contradicts the minimality of E. Therefore, E contains only one element say ν, Since T maps E into itself, ν is a fixed point of T. The uniqueness of the fixed point ν is obvious.

Theorem 1 [7] follows as a corollary of our theorem 2.3. Kirk [9] has proved a fixed poind theorem using theorem of Smulian and the concept of normal structure where, however, the unicity is not guaranteed. The following example shows that our result is indeed an extension of Kirk's theorem,

Example 2.4. Let $X = [0, 7]$ with the usual metric. Define $T : X \to X$ as follows

$$Tx = \begin{cases} x/3 & 0 \leqslant x \leqslant 6 \\ -2x+14 & 6 \leqslant x \leqslant 7 \end{cases}$$

Then T satisfies condition (A) with 0 only fixed point, but T is not nonexpansive. Indeed, let $x=6$, $y=7$ then $d(x, y) = 1 < 2 = d(Tx, Ty)$.

Definition 2.5. Let X be a convex metric space. We will say that X is *strictly convex* if for any x, y in X and $\lambda \varepsilon I$, there exists a unique $z \varepsilon X$ such that $\lambda d(x, y) = d(z, y)$ and $(1-\lambda) d(x, y) = d(z, x)$.

Lemma 2.6. Let X be a strictly convex metric space and C be a nonempty convex subset of X. Let $T:C \to C$ be a mapping as in Theorem 3.2, then $F(T)$ is convex.

Proof. Let x, y be in $F(T)$; we need to show that for any λ, $0 \leqslant \lambda \leqslant 1$; $W(x, y; \lambda)$ belongs to $F(T)$. Since C is convex, $W(x, y; \lambda)$ belongs to C for x, y in $F(T)$ and $\lambda \varepsilon I$. Let $z = W(x, y; \lambda)$. Now

(1) $d(x, y) = d(Tx, Ty) \leqslant d(x, Tz) + d(Tz, y)$.

Using the definition of T we have

(2) $d(x, Tz) \leqslant max \{ d(x, z), [d(x, Tx) + d(z, Tz)]/2$

$[d(x, Tz) + d(z, Tx)]/2 \}$

$= max \{ d(x, z), d(z, Tz)/2, [d(x, Tz) + d(z, x)]/2 \}$

Now $d(x, Tz) \leqslant 1/2\, d(z, Tz) \leqslant 1/2 [d(z, x) + d(x, Tz)]$ implles $d(x, Tz) \leqslant d(x, z)$. Thus from (2) we have

(3) $d(x, Tz) = d(x, T(W(x, y; \lambda)) \leqslant d(x, z) =$

$d(x, W(x, y; \lambda))$;

Similarly we wave

$$(4) \quad d(y, T(W(x, y; \lambda))) \leqslant d(y, W(x, y; \lambda)),$$

Using (3) and (4) we can write (1) as

$$d(x, y) \leqslant d(x, T(W(x, y; \lambda))) + d(T(W(x, y; \lambda)), y)$$

$$\leqslant d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y)$$

$$\leqslant \lambda d(x, x) + (1-\lambda) d(x, y) + \lambda d(y, x) + (1-\lambda) d(y, y)$$

$$= d(x, y).$$

Thus, we conclude that all inequalities are equalities,

i. e.

$$d(x, T(W(x, y; \lambda))) + d(T(W(x,y; \lambda)), y)$$

$$= d(x, W(x,y; \lambda)) + d(W(x, y; \lambda), y).$$

Hence

$T(W(x, y, \lambda)) = W(x, y; \lambda)$. Therefore $F(T)$ is convex.

Definition 3. 1. Let X be a metric space $T: X \to x$ be a mapping. T is said to have *property* (K) on $E \subset X$ *if*, for any closed subset F of E which contains more than one element and which is mapped into itself by T, there exists an $x \in F$ such that

$$d(x, Tx) < \sup_{y \in E} d(y, Ty).$$

Tkeorem 3. 2. Let X be a compact metric space and $T: X \to X$ be a continuous mapping satisfying the following condition :

$$d(Tx, Ty) \leqslant max \{ d(x, y), [d(x,Tx) + d(y, Ty)]/2,$$

$$[d(x, Ty) + d(y, Tx)]/2 \}$$

for all $x, y \in X$. Suppose T has property (K) on X; then T has a fixed point.

Proof. Let H be the family of all nonempty, closed subsets of X which is mapped into itself by T. By Zorn's Lemma it follows that H has a minimal element D, If D consists of a single element, then that element must be a fixed point of T. Suppose D contains more than one element. Since T has property (K) over X, there exists a $z \in D$ such that

$$d(z, Tz) = \gamma < \sup_{y \in D} d(y, Ty),$$

Let $C = \{ x \in D : d(x, Tx) \leqslant \gamma \}$, Then C is nonempty proper subset of D. We claim C is closed too. Indeed, if $\{ x_n \}$ is in C and $x_n \to y$, then $y \in D$ (since D is closed).

By the triangle inequality, $d(y, Ty) \leqslant (y, x_n) + d(x_n, Tx_n) + d(Tx_n, Ty)$. The continuity of T implies $d(y, Ty) \leqslant \gamma$, so y is in C. It remains to show that C is mapped into itself by T. Let $x \in C$, then

$$d(Tx, T^2x) \leqslant max\{ d(xTx), [d(x, Tx) + d(Tx, T^2x)]/2, [d(x, T^2x) + d(Tx, Tx)]/2 \}$$

Hence $d(Tx, T^2x) \leqslant d(x, Tx) \leqslant \gamma$. So C is mapped into itself by T. Thus C is nonempty, closed proper subset of D which is mapped into itself. This contradicts the minimality of D. Hence D consists of a single point, which must be a fixed point of T.

The following examples show that our theorem is indeed a generalization of the theorem of Kannan [7].

Example 3. 3. Let $X = \mathrm{R}$ with the usual metric. Define $T : X \to X$ by $Tx = x$ for all $x \in R$. Then T satisfies condition of our mapping, but not that of Kannan.

Example 3. 4. Let $X = [0, \infty)$ with the usual metric and

$Tx = \frac{x}{1+x}$. Then T satisfies condition of our mapping, but not that

of Kannan. In fact, for $x \neq y$, $d(Tx, Ty) = \frac{(x-y)(xy-1)}{xy} < d(x, y)$,

os T satisfies our condition Let $x = 0$, $y = 1$, then $d(Tx, Ty) =$

$$1/2 > 1/4 = 1/2\,[\,d(x, Tx) + d(y, Ty)\,].$$

Theorem 3. 5. Let X be a strictly convex metric space with property (C). Let D be a nonempty, bounded, closed and convex subset of X. Let $F : D \to D$ be a family of continuous commuting mappings with nonempty fixed points set and satisfying condition of Theorem 3. 2. Then F has a common fixed point in D .

Proof : It follows from Lemma 2 6 that $F(T)$, the fixed points set of T in F, is convex . By the continuity of T we infer that $F(T)$ is closed .

Let F_α be the fixed point set of $T\alpha$ in F ; F_α is nonempty, closed, bounded ard convex . If $u \in F_\alpha$, then for any β, $T_\alpha T_\beta = T_\beta T_\alpha u = T_\beta u$; i. e. $T_\beta u$ lies in F_α and each T_β maps F_α into itself . If we are given a finite sequence $\alpha_1, \alpha_2, \ldots, \alpha_m$, and consider T_{α_m} as a mapping satisfying condition of theorem of $F_{\alpha_1} \cap F_{\alpha_2} \cap \ldots \cap F_{\alpha_m}$ into itself, it follows from the hypothesis that $\bigcap_{k=1}^{m} F_{\alpha_k} \neq \phi$. Since X has property (C), the family $\{F_\alpha\}$ has a nonempty intersection, but this consists of precisely the common fixed points .

Definition. Let B be a bounded set in a metric space X, and let $\delta(B)$ be its diameter. A point $x \in B$ is said to be a diametral point of B if $\sup_{y \in B} d(x, y) = \delta(B)$. A convex subset K of X is said to have normal structure if every bounded convex subset K_1 of K which contains more than one point, has a point that is not a diametral point of K_1 .

Any compact convex metric space has normal structure [11 , Proposition 5] . Every compact convex set of Banach space has a normal strueture [2 p. 6] ; so does every closed bounded and convex subset of a uniformly convex Banach space [10, *p*. 8] .

Corollary 3. 6. Suppose X is strictly convex with property (C). Let K be a nonempty, bounded, closed and convex subset of X with normal structure If F is commuting family of nonexpansive mappings, of K into itself, then the family has a common fixed point .

As our final result, we prove an extension of the results of Goebel, Kirk and Shimi [6] .

Definition 4. 1. Let X be a metric space, $T : X \to X$ be a mapping. T is said to have *property* (S) on $K \subset X$ if for any nonempty, closed bounded convex subset D of K which contains more than one element and which is mapped into Itself by T, we have

$$\sup_{y \in D} d(y; Ty) < \delta (D) ,$$

where δ (D) is the diameter of D.

Definition 4. 2 Let X be a metric space and T: $X \to X$ be a mapping. We say that T satisfies *condition* (GKS) if for all $x, y \in X$ we have

$$d(Tx, Ty) \leqslant a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx),$$

where $a_1, a_4, a_5 \geqslant 0$, $a_2, a_3 > 0$ and $\sum_1^5 a_i = 1$.

Theorem 4.3. Let X be a convex metric space having property (C).

Let K be a nonempty closed, bounded convex subset of X. Let $T : K \to K$ be a mapping staisfying condition (GKS) and having property (S) on X. Then T has a unique fixed point.

Proof : Let F be the family of all nonempty closed bounded convex subsets of K which are mapped into itself. Since $K \in F$, $F \neq \Phi$, using property (C) and Zorn's Lemma we can find a minimal subset D of K in F. Without loss of generality we may assume that $a_2 = a_3$ and $a_4 = a_5$. Let

$$\gamma(D) = (a_1 + a_4 + a_5) \sup_{x,y \in D} d(x, y) + (a_2 + a_3) \sup_{y \in D} d(y, Ty).$$

Then $\gamma(D) < (a_1 + 2a_4)\, \delta(D) + 2a_2\, \delta(D) \leqslant \delta(D)$, since $a_2 > 0$. We claim that D contains one point. Suppose not, let $x, y \in D$, $x \neq y$.

Then,

$$d(Tx, Ty) \leqslant a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty)$$

$$+ a_4 d(x, Ty) + a_5 d(y, Tx)$$

$$\leqslant (a_1 + 2a_4) \sup_{x,y \in D} d(x, y) + 2a_2 \sup_{y \in D} d(y, Ty)$$

$$= \gamma(D).$$

Thus $T(D)$ is contained a closed spherical ball C with center Tx and radius $\gamma(D)$.

It is obvious that $T(D \cap C) \subseteq D \cap C$. By the minimality of D we obtain $D \subseteq C$. Therefore $d(Tx, y) \leqslant \gamma(D)$ for each $y \in D$ and hence

$\sup_{y \in D} d(Tx, y) \leqslant \gamma(D)$ for any arbitrary but fixed $x \in D$. Let

$$D_1 = \{ z \in D : \sup_{y \in D} d(z, y) \leqslant \gamma(D) \}$$

Then D_1 is nonempty, bounded closed and convex. Moreover

$T(D_1) \subset D_1$. Also,

$$\delta(D_1) = \sup_{x,y \in D_1} d(x,y) \leqslant \sup_{\substack{y \in D \\ x \in D_1}} d(x,y)$$

$$\leqslant \sup_{x \in D_1} \sup_{y \in D} d(x,y) \leqslant \gamma(D)$$

$$< \delta(D) .$$

Thus, D_1 is a proper subset of D . This contradicts the minimality of D. Consequently D has only one point which is a fixed point of T in K. The unicity of fixed point is clear .

Remark 4. 4. A similar theorem for uniformly convex Banach was proved by Goebel, Kirk, and Shimi [16] without the restriction that $a_2, a_3 > 0$, however there the continuity of the mapping T was assumed ; moreover the fixed point was not unique ,

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