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ON SOME FIXED POINT THEOREMS

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There has been given a series of extensions of the well-known Banach Contraction Principle. Caristi [1] gave an important theorem where the function is not of contraction type, not even continuous. In this paper we prove a theorem where the function is not necessarily continuous and then we derive several known results as corollaries.

We need the following terminologies.

Let X be a metric space and $f : X \to X$ be a self mapping. Then, for $x \in X$, $O(x,\infty)=(x, fx, f^2x \ldots\ldots)$ is called the orbit of x. X is said to be orbitally complete if each Cauchy sequence in $O(x,\infty)$ converges there. If X is a complete metric space then it is orbitally complete, however, an orbitally complete metric space is not necessarily complete.

Let us recall the following well-known results.

Theorem 1. Let $f : X \to X$ be a contraction map on a complete metric space. Then f has a nnique fixed point.

Theorem 2. Let $f : X \to X$ and $\Phi : X \to [0, \infty)$, where X is a complete metric space and Φ is a lower semicontinuous function. If for each $x \in X$

$$d(x, fx) \leqslant \Phi x - \Phi(fx)$$

then f has a fixed point.

Note that f is not even continuous in this case.

We state our main result as follows .

Theorem 3. Let $f : X \to X$ be a function, where X is an orbitally complete metric space , Assume further that

(i) for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $d(x, fx) < \delta(\varepsilon)$ then $f[B(x, \varepsilon)] \subset B(x, \varepsilon)$, and

(ii) $d(f^n x_0, f^{n+1} x_0) \to 0$ as $n \to \infty$ for some $x_0 \in X$.
Then the sequence $\{f^n x_0\}$ converges to a fixed point of f ,

Proof . Let $0(x, \infty) = \{x, fx, f^2x, \quad .\}$. We want to show that $\{f^n x_0\}$ is a Cauchy sequence .

Given $\varepsilon > 0$ we choose N such that $d(f^n x_0, f^{n+1} x_0) < \delta(\varepsilon)$ for all $n \geqslant N$. For simplicity, we set $x_n = f^n x_0$, for all $n \in N$.

Now $d(x_N, x_{N+1}) = d(x_N, fx_N) < \delta(\varepsilon)$ so we get $f[B(x_N, \varepsilon)] \subset B(x_N, \varepsilon)$. In particular $x_{N+1} \in B(x_N, \varepsilon)$. By induction, we get that $f^i x_N = x_{N+i} \in B(x_N, \varepsilon)$ for all $i \geqslant 0$. Thus $d(x_i, x_r) \leqslant 2\varepsilon$ for $i, r \geqslant N$. This shows that $\{f^n x_0\}$ is a Cauchy sequeuce . Since X is an orbitally complete metric space $\{f^n x_0\}$ converges to some x in X.

We show that $fx = x$.

If $x \neq fx$, then $d(x, fx) = \alpha > 0$. We seek a contradiction .

We choosen sufficiently large such that $x_n = f^n x_0 \in B(x, \alpha/3)$ and $d(x_n, x_{n+1}) < \delta(\alpha/3)$. So $f[B(x_n, \alpha/3)] \subset B(x_n, \alpha/3)$. This gives that $fx \in B(x_n, \alpha/3)$.

However,

$$d(x_n, fx) \geqslant d(fx, x) - d(x, x_n) \geqslant (2/3)\alpha .$$

So $fx \notin B(x_n, \alpha/3)$, a contradiction .

This implies that $d(x, fx) = 0$, *i. e.*, x is a fixed point of f ,

Now, we derive several known results as corollaries .

Remark. If $f : X \to X$ is a contraction map, *i. e.*, $d(fx, fy) \leqslant kd(x, y)$ for any $x, y \in X$, where $0 \leqslant k < 1$, then (i) and (ii) of Theorem 3 are satisfied. Indeed, for any given $\varepsilon > 0$, let $\delta = (1-k)\varepsilon$. Now, if $d(x, fx) < \delta$, then for $z \in B(x, \varepsilon)$ we have that

$$d(fz, x) \leqslant d(fz, fx) + d(fx, x) \leqslant kd(z, x) + d(fx, x)$$
$$< k\varepsilon + (1-k)\varepsilon = \varepsilon$$

Hence (i) is satisfied.

Moreover,

$$d(f^n x, f^{n+1} x) \leqslant k^n d(x, fx) \qquad \text{for any } x \in X.$$

Hence (ii) is satisfied, too.

We close this note with the following result.

Corollary (see [2]). Let $f : X \to X$ satisfy the condition

$$d(fx, fy) \leqslant \Phi(d(x, y))$$

where X is an orbitally complete metric space, $\Phi : R^+ \to R^+$ is a monotone, increasing function such that $\Phi^n(t) \to 0$ for each fixed $t > 0$. Then f has a unique fixed point x, and $x_n = f^n x$ converges to x.

Proof. Clearly $d(f^n x, f^{n+1} x) \to 0$ as $n \to \infty$. Now, since $d(f^n x, f^{n+1} x) \leqslant \Phi^n(d(x, fx))$, let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon - \Phi(\varepsilon)$. If $d(x, fx) < \delta$ then for $z \in B(x, \varepsilon)$ we get $d(x, fz) \leqslant d(x, fx) + d(fx, fz) < \varepsilon - \Phi(\varepsilon) + \Phi(d(x, z))$. Being Φ monotone increasing, we have that $f(z) \in B(x, \varepsilon)$, so $f[B(x, \varepsilon)] \subset B(x, \varepsilon)$.
Then the result follows from Theorem 3.

REFERENCES

[1] **Caristi J.**, Fixed Point Theorems for mappings satisfying inwardness conditions. *Trans. Amer. Math* Soc. **215** (1976), 241-251,

[2] **Matkowski, J.**, Integrable solutions of functional equations, *Diss Math.* CXXVII (1975), 1-68.