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IPER - DUAL SYSTEMS AND HILBERT TRIPLETS

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ABSTRACT

We introduce the concept of iper–dual systems. Beginning from the iper–dual system $[H^*, S>$, we construct the Hilbert triplet $S \subseteq H \subseteq S'$, where S is a pre-Hilbert space dense in the (real or complex) Hilbert space H and S' is a space of linear functionals on S wider that H. The couple $[S', S>$ so obtained is an iperdual system. Once having introduced the order-bounded nets, we prove that if every weak-convergent net is order–bounded, then $S=H$. Finally we study four examples of special triplet $S \subseteq H \subseteq S'$.

0. INTRODUCTION

In Section 1 we introduce the concept of iper-dual systems, as a class of couples of linear spaces placed between the class of dual systems [1], and the class of Hilbert spaces.

We construct a space of linear functionals X' by means of the X–weak Cauchy sequences so that $X \subseteq \hat{X} \subseteq X'$, where $[\hat{X}, X>$ is a iper-dual system.

In Section 2 we consider the iper-dual system $[H^*, S>$, where S is a pre-Hilbert space, H is the Hilbert space completion of S and H^* is the dual of H. We construct the triplet $S \subseteq H \subseteq S'$ from $[H^*, S>$, in such a way that $[S', S>$ is an iper–dual system, too. The main result obtained in this section is that if every weak convergent net is order-bounded, then $S=H$.

Finally in Section 3 we study four examples of triplet $S \subseteq H \subseteq S'$, In the exemple 1, H is a separable, complex Hilbert space with a fixed orthonormal basis $\{u_n\}$, $S := Sp(\{u_n\})$ is the linear manifold spanned by $\{u_n\}$ and S' is constructed as descripted in Section 1. In this context we show that S' is identifiable with C^N, the space of all complex sequences.

In the example 2, $H = L_2(R^n)$ and S is the space $\mathrm{S}(R^n)$ of functions of rapid decrease in R^n. In this case S' coincides with the space of temperated distributions.

In the example 3, $H = l_2$ and $S = \left\{ \{a_n\} : \sum_{n=1}^{\infty} | a_n|^2 (n+1)^m < \infty \; \forall m \in N \right\}$.

We show that $S' = \{ \{b_n\} : \exists c \in R$ *and* $\exists m \in N$ *such that* $|b_n| \leqslant c(1+n)^m$ $\forall n \in N\}$, who coincides with the dual of S with respect to the topology of conutably normed space that makes S isomorphic to $\mathrm{S}(R)$.

In the example 4, H is the space of Hilbert- Schmidt operators on a separable Hilbert space **H** and S is the space of finite rank operators on **H**. If we denote with $K(\mathbf{H})$ and $\mathbf{B}(\mathbf{H})$ the space of compact operators and the space of bounded operators on **H** resp ectively, we show that $\mathbf{K}(\mathbf{H}) \subseteq S' \subseteq \mathbf{B}(\mathbf{H})$.

It is useful to observe that in all previous examples, S is not complete in the topology induced by inner product. The fact that S is not complete is the necessary and sufficient condition in order that we obtain an *actual* enlargment S' of H^*.

1· IPER-DUAL SYSTEMS AND WEAK CAUCHY SEQUENCES

Let $\hat{X}, X$ be two linear spaces. We suppose that :

(IDS-1) X is a linear manifold of $\hat{X}$.

$(IDS\text{-}2)$ There is a functional $[,>:\hat{X}\times X\rightarrow C$ such that

(a) the restriction of $[,>$ to $X\times X$ is an inner product on X denoted by $<|>$.

(b) $[\hat{x},\,.\,>$ is a linear functional for every $\hat{x}\in\hat{X}$

(c) $[.,x>$ is an antilinear functional for every $x\in X$.

(d) $[\hat{x},x>=0 \;\; \forall x\in X \Rightarrow \hat{x}=0$.

Set $\bar{R}=\mathbb{R}\cup\{+\omega\}$. We define the function (called infi-norm) $\|\;\|_{\wedge}:\hat{X}\rightarrow\bar{R}$ in the following manner :

$$\|\hat{x}\|_{\wedge}=sup\{\,|\,[\hat{x},y>\,|\,:\,y\in X,\,\|y\|=1\}$$

where, of course, $\|y\|^2=<y|y>$. Then we have

$(N\text{-}1)$ $\|\hat{x}\|_{\wedge}=0 \;\; iff \; \hat{x}=\underline{o}$

$(N\text{-}2)$ $\|\lambda\hat{x}\|_{\wedge}=|\lambda|\,\|\hat{x}\|_{\wedge} \quad \forall\lambda\in C$

$(N\text{-}3)$ $\|\hat{x}+\hat{y}\|_{\wedge}\leqslant\|\hat{x}\|_{\wedge}+\|\hat{y}\|_{\wedge}$

We remark that the generalized Schwartz inequality holds :

$$|\,[\hat{x},x>\,|\leqslant\|\hat{x}\|_{\wedge}\,\|x\| \quad \forall\hat{x}\in\hat{X},\;\forall x\in X.$$

(of course, $\|x\|_{\wedge}=\|x\| \;\forall x\in X$) .

DEFINITION.

We say that $[\hat{X},X>$ is an iper-dual system if (IDS-1), (IDS-2) hold and moreover

(IDS-3) $\bar{X}\subseteq\hat{X}$, where $\bar{X}$ is the Hilbert space completion of

X with respect to the inner prduct $<1>$.

Let $[\hat{X},X>$ be an iper-dual system. We say that a sequence $\{\hat{x}_n\} \subseteq \hat{X}$ is X-weak Cauchy *iff* $\exists \lim_n [\hat{x}_n, y> \ \forall y \in X$.

We say $\{\hat{x}_n\}$ is X-weak convergent to $\hat{x} \in \hat{X}$ *iff* $\lim_n [\hat{x}_n, y> = [\hat{x}, y> \ \forall y \in X$.

We see immediately that the criterion of weak convergence defined on X is compatible with its linear structure [2].

Furthermore, we observe that there exist X-weak convergent sequences for which the corresponding sequences of infi-norms do not converge to the infi-norm of limit.

Let $L_w(\hat{X}, X)$ denote the linear space of the X-weak Cauchy sequences in $\hat{X}$. Every X-weak Cauchy sequence $\{\hat{x}_n\}$ defines a linear functional $t : X \rightarrow C$ by the law $<t,x> := \lim_n [\hat{x}_n, x> \ \forall x \in X$.

The set of all linear functionals generated by some X-weak Cauchy sequence is denoted by X'. In the same way, generalizing the notion of X-weak Cauchy sequence, we shall say that a sequeuce $\{t_n\}$ in X' is a X-weak Cauchy sequence *iff* $\exists \lim_n <t_n, x> \ \forall x \in X$.

Any X-weak convergent sequence in X' is a X-weak Cauchy sequence too. Now let $\{t_n\}$ be a X-weak Cauchy sequence in X', then we can define the linear functional $l : X \rightarrow C$ *as* $<l,x> = \lim_n <t_n,x> \ \forall x \in X$ and we write $l = w-\lim t_n$.

In general l is not an element of X since it is not assured that there exists a sequence $\{\hat{x}_n\}$ in $\hat{X}$ which converges X-weak to l. In the next proposition we shall give a sufficient condition ensuring that $l \in X$:,

PROPOSITION 1. *Let* $\{t_n\}$ be a X-weak Cauchy sequence in X'.

If any t_n is generated by a sequence $\{\hat{x}_j^{(n)}\}_{j\in N}$ in $\hat{X}$ which is uniformly weak convergent to t_n in the sense that for every $n\in N$ and every $y\in X$ there exists an integer $j=j(n)$ depending on n but not on y, such that

$$|<t_n, y>-[\hat{x}_{j(n)}^{(n)} \, , y>| \leqslant \frac{1}{n} \quad \forall y \varepsilon X$$

then $l=w\text{-}lim\ t_n$ is an element of X'.

PROOF. Considering the sequence $\{\hat{x}_{j(n)}^{(n)}\}$ in $\hat{X}$ we get that it is a X-weak Cauchy sequence; indeed, from

$$|[\hat{x}_{j(n)}^{(n)} \, , y> -[\hat{x}_{j(m)}^{(m)} \, , y>| \leqslant |<t_n, y> - [\hat{x}_{j(n)}^{(n)} \, , y>|+$$

$$|<t_n \, , y> - <t_m \, , y>|+|<t_m, y> - [\hat{x}_{j(m)}^{(m)} \, , y>|$$

it follows that $\exists\ lim\ [\hat{x}_{j(n)}^{(n)} \, , y> \ \forall y\epsilon X$.

On the other hand, the relation

$$| <l, y>-[\hat{x}_{j(n)}^{(n)} \, , y>| \leqslant |<l, y>-<t_n,y>|+|<t_n, y>-[\hat{x}_{j(n)}^{(n)} \, , y> |$$

under our hypotheses gives the result $l\epsilon X'$.

On $L_w(\hat{X}, X)$ we can introduce the following equivalence relation, called the X-weak equiconvergence relation, defined as follows :

$$\{\hat{x}_n'\} \sim_w \{\hat{x}_n''\} \ iff \lim_n [\hat{x}_n', y> = \lim_n [\hat{x}_n'', y> \quad \forall y\epsilon X.$$

It is trivial to see that X' and $L_w(\hat{X}, X)/\sim_w$ are linearly isomorphic. Set $t\equiv[\{\hat{x}_n\}]\sim_w$. We suppose that $\|\hat{x}_n\|_\wedge<M$. If the sequence $\{y_k\}\subseteq X$ is norm-convergent to zero, then

$$0\leqslant|<t, y_k>|=\lim_n|[\hat{x}_n,y>|\leqslant\sup_n\|\hat{x}_n\|_\wedge\cdot\|y_k\|\leqslant M\|y_k\|$$

so that $\lim_k<t,y_k>=0$. Thus t is sequentially continuous (and therefore continuous) with respect to the topology on X induced by the inaer product. Hence there exists $x_0\in\bar{X}\equiv\bar{X}^*$ such that $t=x_0$, where $\bar{X}^*$ denotes thc dual space of $\bar{X}$ with respect to the topology induced by the inner product. Therefore we have proved that :

PROPOSITION 2.

If we denote by $L^b_w(\hat{X},X)$ the linear subspace of $L_w(\hat{X}, X)$ of bounded sequences in infl-norm, then $\bar{X}$ and $L^b_w(\hat{X},X)/\sim_w$ are linearly isomorphic. In particular, it follows that

$$\|x\|_\wedge<\infty\ \forall x\in\bar{X} \text{ and } \|\hat{x}\|_\wedge=+\infty\ \forall\hat{x}\in\hat{X}\setminus\bar{X}$$

2. HILBERT TRIPLETS AND WEAK CONVERGENT NETS

We shall construct now an iper–dual system by means of an other iper–dual system. In this Section, we denote by S a pre–Hilbert spa e with inner product $<1>$. $(S, T_{<1>}(S))$ denotes the locally convex topological vector spoce in which $T_{<1>}(S)$ is the topology induced by the inner product. S^* denotes the dual space of $(S, T_{<1>}(S))$ and H the Hilbert space completion of S. We denote by $D(S)$ the linear space antiisomorphic to S of all linear functionals $<x| : S\to C$, with $x\in S$, defined by $<x| : y\to<x|y>\ \forall y\in S$. We recall that $S=H$ iff $D(S)=S^*$. $[H^*,S>$ is an iper-dual system. We construct S′ in the same manner as in the previous Section :

$$S' = \{t : S \to C : \exists \{x_n\} \in L_w (H^*, S), t = w\text{-}\lim_n \langle x_n | \} .$$

We have that $S' \equiv L_w(H^*, S)/\sim_w$ and we have obtained the triplet

$$S \leqslant H \equiv H^* \equiv S^* \subseteq S'$$

$[S', S\rangle$ is an iper-dual system .

REMARK 1. It is easy to prove that the Hilbert space H^* is identifiable with $L_w^b (D(S), S)/\sim_w$ in which bounded S-weak Cauchy sequences *from* $D(S)$ are involved. We shall remember that a standard procedure for constructing the completion H^* of $D(S)$ is based on strong Cauchy sequences using the strong equiconvergence relation

$$\{x_n'\} \sim_s \{x_n''\} \text{ iff } \lim_n \|x_n' - x_n''\| = 0 .$$

The above quoted result could seem very surprising, since S-weak Cauchy sequences are "much more" than strong Cauchy sequences from $D(S)$. However, it must be remarked that the S-weak equiconvergence relation is weaker than the strong equiconvergence relation, so that S-weak equiconvergence classes are "larger" than strong equiconvergence classes .

REMARK 2. If $\{x_n\}$ is a sequence from H* which is norm-bounded, then the existence of the $\lim_n \langle x_n | y \rangle \; \forall y \in S$ implies the existence of the $\lim_n \langle x_n | t \rangle \; \forall t \in H$. However, the conditions

(i) $\{x_n\} \sim_w \{x_n'\}$

(ii) $\exists \lim_n \langle x_n, t \rangle \; \forall t \in H$

(iii) $\|x_n\| \leqslant M$

do not imply either the norm boundedness of $\{x_n'\}$ or the existence of the $\lim_n <x_n|t> \ \forall t \in H$.

We can endow the space S^* with the $\sigma(S^*, S)$–topology generated by the basis of neighbourhoods of zero

$$U(x_1, \ldots, x_n; \varepsilon) = \{l \in S^* : |<l, x_j>| < \varepsilon \ \forall j = 1, \ldots, n\},$$

$$x_1, \ldots, x_n \in S, \ \varepsilon \in R_+$$

so in this way $(S^*, \sigma(S^*,S))$ is an Hausdorff locally convex space whose topological dual is S. We recall that $\sigma(S^*,S)$ is the weakest topology for S^* with respect to which all the element of S are continuous. Hence we have that $\sigma(S^*,S) \leqslant T_{<1>}(S^*)$.

The most important properties of the Hilbert trip'et $S \subseteq H \equiv H^* \subseteq S'$ in relation to the corresponding iper-dual system ore sumarized in the following statements :

(a) $(S, \sigma(S, S'))$ *is a nuclear space* ;

(b) *the canonical embedding* $i : (S, \sigma(S,S')) \rightarrow (H, \sigma(H, H^*))$ *is continuous;*

(c) H^* *is dense in* $(S', \sigma(S',S))$.

PROOF. (a) The topology $\sigma(S, S')$ for S is the projective topology with respect to the family $\{(C_\alpha, L_\alpha) : C_\alpha = C, L_\alpha S\}$. Since each $C_\alpha = C$ is nuclear, then the space $(S, \sigma(S, S'))$ is nuclear too [3] ,

(c) We shall make the identification $H \equiv H^*$. Let $t \in S'$. Then $t = w\text{-}\lim_n x_n$, $\{x_n\} \in L_w(H^*,S)$. We shall define $t_k := w\text{-}\lim_n <x_n^{(k)}|$ where $\{x_n^{(k)}\}$ is the sequence in $L_w^b(H^*,S)$ defined by $\{x_1, x_2, \ , x_k, x_k, x_k, \ldots\}$. It results that $<t_k|, y> = <x_k|y>$ and so

$$\lim_k <t_k, y> = <t, y> \ \forall y \in S.$$

For sake of completeness we give now briefly some definitions and results useful in the sequel .

A partially ordered set I is said to be directed if, given $\alpha,\beta\in I$ there exists $\gamma\in I$ such that $\alpha\in\gamma$ and $\beta\leqslant\gamma$.

A generalized sequence or a net in a topological space T is a mappiag $f: I\to T$, where I is a directed set. As in the case of sequences, we write $f(\alpha)=x_\alpha$ and we represent the net by its image $\{x_\alpha\}$.

Given the net $\{x_\alpha\}$ in the topological vector space T, we say that $x\in T$ is the limit of $\{x_\alpha\}$ (and we write $x=lim\ x_\alpha$) or that x_α converges to x if for any neighbourhood U_x of x there exists $\alpha_0\in I$ such that $x_\alpha\in U_x\ \forall\alpha\geqslant\alpha_0$. It is well known that a function f from a topological space S to a topological space T is continuous iff for every convergent net $\{x_\alpha\}$ in S, with $x_\alpha\to x$, the net $f(x_\alpha)$ converges in T to $f(x)$. A net $\{x_\alpha\}$ in S is said to be order-bounded if there exists $\alpha_0\in I$ snch that $\{x_\alpha : \alpha\geqslant\alpha_0\}$ is norm-bounded .

The main result of this section is the following

THEOREM. (a) $S=H \Rightarrow L_w(H^*,S) = L_w^c(H^*, S)$

(b) If every net $\sigma(S, D(S))$ - convergent is order-bounded, then $S=H$.

PROOF. (a) It is an obvious consequence of the Uniform Boundedness Principle [4] .

(b) It is sufficient to prove that $D(S)=S^*$. Since the topological dual of $(S, \sigma(S, D(S)))$ is $D(S)$, it is sufficient to show that every $l\in S^*$ is $\sigma(S, D(S))$ - continuous on S. For this, let $l\in S^*$ and let $\{x_\alpha\}$ be a net $\sigma(S, D(S))$ - convergent to x . Since $l\in S^*$, there exists $\{y_k\}\subseteq D(S)$ such that $\|L-y_k\| \underset{k\to\infty}{\to} 0$. Therefore

$$|<l, x_\alpha-x>|=|<l\pm y_k, x_\alpha-x>|\leqslant \|l-y_k\| (\|x_\alpha\|+\|x\|)+|<y_k| x_\alpha-x>| .$$

Since $\{x_\alpha\}$ is order-bounded, there exist an $\alpha_0\in I$ and a real number $M>0$ such that $\|x_\alpha\|<M$ for every $\alpha\geqslant\alpha_0$. So, $\varepsilon>0$, there exists $k_0\in N$ such that $\|y_{k0}-l\| <\varepsilon/2\ (M+\|x\|)$. Moreover, as $\{x_\alpha\}$ is $\sigma(S, D(S))$

- convergent to x, there exists $\bar{\alpha}\in I$ such that $|<y_{k0} \mid x_\alpha - x>| < \varepsilon/2$ for every $\alpha \geqslant \bar{\alpha}$.

Let $\alpha_1 \geqslant \{ \alpha_0, \bar{\alpha}\}$. Then for every $\alpha \geqslant \alpha_1$ we have

$$|<l, x_\alpha - x>| \leqslant \|l - y_{k0}\|(\|x_\alpha\| + \|x\|) + |<y_{k0}|x_\alpha - x>| < \frac{\varepsilon}{2(M+\|x\|)}(H + \|x\|)$$

$$+ \frac{\varepsilon}{2} = \varepsilon, \text{ i.e. } \lim <l, x_\alpha> = <l, x> .$$

REMARK. The statment (b) has a converse when H is finite-dimensional (i. e. in a finite-dimensional Hilbert space every weakly convergent net is order-bounded), but the author ignores whether this property characterizes any Hilbert space .

3. EXAMPLES

EXAMPLE 1. We denote by H a separable, infinite-dimensional complex Hilbert space, by $\{u_n\}$ an orthonormal basis fixed in H and by $S = Sp(\{u_n\})$ the linear manifold spanned by $\{u_n\}$.

Let $E_k : H \to H$ be the projection on the finite-dimensional subspace $S^k := Sp(u_1, \ldots, u_k)$:

$$E_k x := \sum_{j=1}^{k} <u_j \mid x> u_j \quad \forall x \in H .$$

It is straightforward to verify that the functional $<1>_k : S \times S \to C$ defined by

$$<x| y>_k := <E_k x \mid E_k y> = \sum_{j=1}^{k} <x \mid u_j><u_j \mid y>$$

is a degenerate inner-product on S. We shall denote S_k the space S endowed with the degenerate inner product $<1>_k$.

The function $p_k : x \rightarrow p_k(x) := \left(\sum_{j=1}^{k} |<u_j | x>|^2 \right)^{1/2}$ is a seminorm on S.

Of course for the family of seminorms $F(\{u_n\}) = \{p_k : k \in N\}$ holds $p_k(x) \leqslant p_{k+1}(x) \; \forall k \in N$ and $\forall x \in S$. In the previous Section we have endowed the pre-Hilbert space S with the weak topology $\sigma(S, D(S))$, that is of the topology generated by the family of seminorms $F(D(S)) = \{q_k := |<x | .>|, k \in S\}$.

We show now that in the present example we have that

$F(D(S))$ is equivalent to the totally ordered family of seminorms $F(\{u_n\})$.

PROOF. For every q_x, $x \in S$, we can write $x = \sum_{j=1}^{n} a_j u_j$ and so for every $y \in S$,

$$q_x(y) = |< \sum_{j=1}^{n} a_j u_j | y>| \leqslant \sum_{j=1}^{n} | a_j | \, | <u_j | y>| .$$

If $A = max(|\alpha_j|)$, we have that $q_x(y) \leqslant A \sum_{j=1}^{n} |<u_j | y>| \leqslant A \sum_{j=1}^{n} p_j(y)$.

Moreover, $(p_k(x))^2 = \sum_{j=1}^{k} | <u_j | x> |^2 \leqslant \left(\sum_{j=1}^{k} | <u_i | x>| \right)^2$

$$= \left(\sum_{j=1}^{k} q_{u_j}(x) \right)^2 \Rightarrow p_x(x) \leqslant \sum_{j=1}^{n} q_{u_j}(x) .$$

Finally, we construct S'. We recall that

$$S' = \{ t : S \rightarrow C : \exists \{x_n\} \in L_w(H^*, S) \text{ such that } <t, y> = \lim_n <x_n | y> \; \forall y \in S\} .$$

Since $S = Sp(\{u_n\})$, we have that

Let $t \in S'$. If $a_j = \langle t | u_j \rangle \ \forall j \in N$ and $y \in S$, $y = \sum_{i=1}^{r} \lambda_i u_i$, then

$$\langle t, y \rangle = \sum_{i=1}^{r}{}' \lambda_i a_i .$$

The (*)[1] suggest to put, *formally*, $t = \sum_{j=1}^{\infty} a_j u_j$, so we can identify t with the sequence $\{a_j\}$.

The map $\phi : S' \to C^N$, *defined for every* $t = \sum_{j=1}^{\infty} a_j u_j$ *in* S' *by* $\phi(t) = \{a_j\}$ *is a linear isomorphism between* S' *and* C^N.

In fact it is easy to verify that ϕ is linear and injective. If $\{a_j\} \in C^N$ we consider the sequence $\{x_n\} \subseteq S$ defined by $x_n = \sum_{j=1}^{n} a_j u_j$. It is easy to see that $\{x_n\} \in L_w(H^*, S)$ and if $t = w\text{-}\lim_n x_n$, *then* $\phi(t) = \{a_j\}$, *thus* ϕ is surjective.

EXAMPLE 2. If H is the Hilbert space $L_2(R^n)$ of square Lebesgue-integrable complex-valued functions on R^n and S is the space $T(R^n)$ dense in $L_2(R^n)$ of the functions of rapid decrease on R^n, then S is the space of temperated distributions $T'(R^n)$ according to the definition in [5]. We have so obtained the iper-dual system $[T'(R^n), T(R^n)\rangle$ and the Hilbert triplet $T(R^n) \subseteq L_2(R^n) \subseteq T'(R^n)$.

EXAMPLE 3. *Let* $H = l_2$ and $S = T$, where

$$T = \{ \{a_n\} \in l_2 : \Sigma \mid a_n \mid^2 (n+1)^m < \infty \ \forall m \in N \} .$$

We show that S' coincides with the pual of T endowed with a suitable topology. If $m \in N$ is fixed, we define the norm $\| \ \|_m$ with

$$\| \underline{a} \|_m := (\Sigma \mid a_n \mid^2 (n+1))^{1/2} \ \forall \ \underline{a} = \{a_n\} \in T .$$

The family of norms $(\| \|_m)_{m \in N}$ is a directed family of norms, that is $\forall k_1, \ldots, k_r \in N$, $\exists\, k \in N$ and $\exists\, c \in R_+$ such that $\| \underline{a} \|_{k_1} + \ldots + \| \underline{a} \|_{k_r} \leqslant c \| \underline{a} \|_k \ \forall \underline{a} \in T$ since, put $k = max\ (k_1, \ldots, k_r)$, we have $\| \underline{a} \|_{k_1} + \ldots + \| \underline{a} \|_{k_r} \leqslant r \| \underline{a} \|_k$.

We consider now the countably normed space $(T, \| \|_m)$. It is well known that a sequence $(\underline{a}^{(n)})_{n \in N}$ in $(T, \| \|_m)$ converges to zero iff $\lim_n \| \underline{a}^{(n)} \|_m = 0 \ \forall m \in N$. It is also easy to prove that $(T, \| \|_m)$ is a complete countably normed space.

The proof of following lemmas 1 and 2 are in [6] :

LEMMA 1. A linear functional $t : (T, \| \|_m) \rightarrow C$ is continuous iff $\exists c \in R_+$ and $\exists k \in N$ such that

$$| t (\underline{a}) | \leqslant c \| \underline{a} \|_k \quad \forall \underline{a} \in T.$$

LEMMA 2. (UNIFORM BOUNDEDNESS PRINCIPL IN COUNTABLY NORMED SPACES). Let X be a complete countabl normed space with $\| \|_r$ a directed sequence of norms Let **F** be a s in X^*, the dual of X. If $\{ F ((f) : F \in \mathbf{F} \}$ is bounded for each f X then there is a $c \in R_+$ and an $r \in N$ so that, for all $f \in X$ and all F **F**,

$$| F (f) | \leqslant c \| t \|_r .$$

Utilizing the Lemmas 1 and 2 it is now easy to prove the

LEMMA 3. Let $(\underline{x}^{(n)} \subseteq l_2$ be a sequence in l_2 such that $\exists \lim_n < \underline{x}^{(n)} | \underline{a} > \ \forall \underline{a} \in T$. Then, if $t = w\text{-}\lim_n < \underline{x}^{(n)} |$, we have that $t \in (T, \| \|_m)'$, where $(T, \| \|_m)'$ is the topological dual of the countably normed space $(T, \| \|_m)$.

LEMMA 4. Let $(\underline{e}^{(n)})$ be the canonical basis in l_2 $(\underline{e}^{(n)} = \{ \delta_{kn} \}_{k \in N})$.

Let $T \in (\mathbf{T}, \| \|_m)'$, $b_n = T(\underline{e}^{(n)})$. Then there is is a $c \in \mathbb{R}$ and a $k \in N$ so that $| b_n | \leqslant c(1+n)^m$ and

$$(*)^3 \quad T\underline{a} = \sum_{n=1}^{\infty} a_n b_n \ \forall \ \underline{a} \in \mathbf{T},$$

Conversely, if $\| b_n \| \leqslant c(1+n)^m$, then $(*)^3$ defines an element of $(\mathbf{T}, \| \|_m)'$.

PROOF. If $T \in (\mathbf{T}, \| \|^3)'$, then from lemma 1 follows $| b_n | \leqslant c(1+n)^m$ and moreover the $(*)^3$ is trivial, Conversely, if $| b_n | \leqslant c(1+n)^m$ then

$$| \Sigma a_n b_n |^2 \leqslant (\Sigma | a_n |^2 (n+1)^{2m+2})(\Sigma | b_n |^2 (n+1)^{-(2m+2)})$$

$$\leqslant 1/6 \, c^2 \pi^2 \, \| \underline{a} \|^2_{2m+2}$$

and so, still from lemma 1, the $(*)^3$ defines an element of $(\mathbf{T}, \| \|_m)'$

THEOREM. In the previous notations, we have $S' = (\mathbf{T}, \| \|_m)'$,

PROOF. $S' \subseteq (\mathbf{T}, \| \|_m)'$ is the Lemma 3.

Conversely, if $T \in (\mathbf{T}, \| \|_m)'$, then from Lemma 4, if $b_n = T(\underline{e}^{(n)})$, it results $| b_n | \leqslant c(1+n)^m$ and $T(\underline{a}) = \Sigma a_n b_n$. But then, put $\underline{x}^{(n)} = (b_1, \ldots, b_n, 0, 0, \ldots)$ we have that $\forall \underline{a} \in T$, $\exists \lim_n \langle \underline{x}^{(n)} | \underline{a} \rangle = \Sigma b_n a_n = T(\underline{a})$

We have thus obtained the iper-dual system $[(\mathbf{T}, \| \|_m)', \mathbf{T}\rangle$ and the Hilbert triplet $\{\{a_n\} : \Sigma | a_n |^2 (n+1)^n < \infty \ \forall m\}$ $\subseteq \{\{a_n\} : \Sigma | a_n |^2 < \infty\} \subseteq \{\{a_n\} : \exists \, c \in R_+, \exists \, m \in N, | a_n | \leqslant c(1+n)^m\}$.

REMARK. If $(\mathbf{T}(R), \| \|_{b,h})$ denotes the countably normed space of functions of rapid decrease on R with $\| \|_{k,h}$ defined by $\|\Phi\|_{k,h} := \| x^k \Phi^{(h)}(x) \|_\infty$, where $\| \Phi \|_\infty := \sup_{x \in R} | \Phi(x) |$, it is well known that the countably normed space ($\mathbf{T}(R)$, $\| \|_{k,h}$) and ($\mathbf{T}$, $\| \|_m$) are isomorphic ([6]). This provide, "en passant", a proof of equivalence of definitions of tempered distributions via linear functionals ([f]) and via weakly Cauchy sequences [5] .

EXAMPLE 4. Let H be a separable Hilbert space , Let **B** (H) be the space of bounded operators on **H**. We are interested to the following linear varieties of **B** (H) :

$$\mathbf{F}\ (\mathbf{H}) := \{ \text{ finite rank operators } \}$$
$$HS\ (\mathbf{H}) := \{ \text{ Hilbert-Schmidt operators } \}$$
$$\mathbf{K}\ (\mathbf{H}) := \{ \text{ compact operators } \}$$

We recall that [8] :

(1) $F \in \mathbf{F}(\mathbf{H})$ iff the range $R\ (F)$ of F is finite-dimensional .

(2) $K \in \mathbf{K}(\mathbf{H})$ iff $\exists \{ v_l \}$, $\{w_l\}$ orthonormal bases of **H** and $\exists \{\mu_l\} \subseteq R_+$ with $\mu_1 \geqslant \mu_l \ \forall l \in N$ and $\lim_l \mu_l = 0$, so that

$$(*)_1^4 \qquad Kx = \sum_{l=1}^{\infty} \mu_l < x \mid v_l > w_l \quad \forall x \in \mathbf{H} .$$

(3) $T \in HS\ (\mathbf{H})$ iff $T \in \mathbf{K}(\mathbf{H})$ and the sequence $\{\mu_l\}$ of $(*)_1^4$ is in l_2 ,

(4) $\mathbf{T}\ (\mathbf{H}) \subset HS\ (\mathbf{H}) \subseteq \mathbf{K}(\mathbf{H}) \subseteq \mathbf{B}\ (\mathbf{H})$.

(5) If $T_1, T_2 \in HS\ (\mathbf{H})$ and $\{u_l\}$ is an orthonormal basis of **H**, the numerical series $\sum_{l=1}^{\infty} < T_1 u_l \mid T_2 u_l >$ is absolutely convergent and the sum is independent from the orthonormal basis $\{u_l\}$.

(6) $HS\ (\mathbf{H})$ can be endowed with the inner product $< T_1 \mid T_2 > := \Sigma < T_1 u_l \mid T_2 u_l >$ where $\{u_l\}$ is any orthonormal basis in **H**. With respect to this inner product, $HS\ (\mathbf{H})$ is a separable Hilbert space, which we denote by H.

(7) If we denote with S the space F(H), it results that S is dense in H. So we can construct S' as in Section 2 .

On the contrary of example 1 in which beginning from the space S of finite sequences we have obtained that S is the space of *all* sequences, now, beginning from the space S of finite rank operators, we obtain that $S' \subset$B (H) and so it is not made up of *all* linear operators on **H**, as it results from the following theorem :

THEOREM In the previous notations, we have $K(H) \subseteq S' \subseteq B(H)$.

PROOF. Let $K \in K(H)$. Then $Kx = \sum_{1}^{\infty} \mu_l <x|v_l> w_l$, $\forall x \in H$. We can look to K as a functional on S defined by

$$K(F) := = \sum_{l=1}^{s} \lambda_l < K \tilde{v}_l \,|\, \tilde{w}_l > \forall F \in S,\ Fx = \sum_{l=1}^{s} \lambda_l <x \,|\, \tilde{v}_l> \tilde{w}_l \ .$$

For each $n \in N$ we define $K_n \in H$ with

$$K_n X = \sum_{l=1}^{n} \mu_l <x \,|\, v_l> w_l \ ,$$

and we show that $K = \text{w } lim < K_n \,|$.

Indeed if $F \in S$, $Fx = \sum_{l=1}^{s} \lambda_l <x \,|\, \tilde{v}_l > \tilde{w}_l$,

$$(*)_2^4 \quad <K_n \,|\, F> = \sum_{l=1}^{\infty} <K_n \tilde{v}_l \,|\, F \tilde{v}_l > = \sum_{l=1}^{s} \lambda_l < K_n \tilde{v}_l \,|\, \tilde{w}_l >$$

and so $\lim_n <K_n | F> = K(F)$. In this way we have proved that$K(H) \subseteq S'$.

We show now that $S' \subseteq B(H)$. Let $t = w\text{-}\lim_n < T_n \in HS$ (H) . From $(*)_2^4$

it is easy to see that

$(*)^4_3 \quad \exists \lim_n < T_n x \mid y > \ \forall x, y \in \mathbf{H}$

(in effect this is a necessary and sufficient condition in order that $\exists \lim_n < T_n \mid F > \ \forall F \in S$).

Of course the $(*)^4_3$ is equivalent to

$(*)^4_4 \quad \exists \lim_n < T_n^* x \mid y > \quad \forall x, y \in \mathbf{H}$

(we recall that $T \in HS(\mathbf{H}) \Rightarrow T^* \in HS(\mathbf{H})$. where T^* denotes the adjoint of T) .

Now from $(*)^4_3$, once fixed x, from the Uniform Boundedness Principle and from the Fréchet–Riesz theorem, it follows that $\exists ! Tx \in \mathbf{H}$ such that

$$\lim_n < T_n x \mid y > \; = \; < T x \mid y > \quad \forall y \in \mathbf{H} .$$

As x run over $\mathbf{H}$, we obtain a map $T : \mathbf{H} \to \mathbf{H}$ which is linear, as it is easy to verify .

Analogously from $(*)^4_4$ we obtain a linear mapping T satisfying

$(*)^4_5 \quad \lim_n < T_n^* x \mid y > \; = \; < \tilde{T} x \mid y > \quad \forall x, y \in \mathbf{H}$

There remain to prove that $T \in \mathbf{B}(\mathbf{H})$. We recall that is sufficient to prove that T has an adjoint operator defined *on* $\mathbf{H}$, [4] .

This is immediately verified once noted that the adjoint of T is the operator $\tilde{T}$ since $< Tx \mid y > \; = \lim_n < T_n x \mid y > \; = \lim_n < x \mid T_n^* y >$

$= \; < x \mid \tilde{T} y > \ \forall x, y \in \mathbf{H}$.

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