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GENERALIZED FIXED POINT THEOREMS IN METRIC SPACE

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ABSTRACT

B. E. Rhoades proved several fixed point theorems for mapping satisfying much more general contractive type conditions for three points than those of pittnauer (1975). They have taken only single operator. In this paper we proved fixed point theorems involving three and four points and at the same time two and more operators. These theorems are more general than those of pittnauer and Rhoades

Theorem 1.

Let T_1 and T_2 be two operators mapping on a complete metric space X, with metric d into itself such that $T_1^p T_2^q(x) = T_2^q T_1^p(x)$, for $p, q \geqslant 1$ and $T_1^p T_2^q(x)$ means T_1 operating p times and T_2 operating q times. Further if for a real number h satisfying
$0 \leqslant h \leqslant 1$ and for each $x, y, z \in X$,

$$d(T_1(x), T_2(y)) \leqslant h \max \{d(x,y), d(x,z), d(y,z),$$

$$d(x, T_1(x)), d(y, T_2(y)), d(x, T_2(y)),$$

$$d(y, T_1(x)), [d(x, T_1^p T_2^q(z)) + d(y, T_1^p T_2^q(z))],$$

$$\frac{[d(x, T_1^p T_2^q(z)) + d(T_1(x), T_1^p T_2^q(z))]}{2},$$

$$\frac{[d[y, T_1^p T_2^q(z)) + d(T_2(y), T_1^p T_2^q(z))],}{2},$$

$$\frac{[d(x, T_1^p T_2^q(z)) + d(T_2(y), T_1^p T_2^q(z))]}{2},$$

$$\frac{[d(y, T_1^p T_2^q(z)) + d(T_1(x), T_1^p T_2^q(z))]\}}{2}, \quad . \qquad (1)$$

Then T_1, T_2 have a unique common fixed point.

Proof.

Let $x_0 \in X$ and consider a sequence $\{x_n\}$ by setting

$$x_1 = T_1(x_0), x_2 = T_2(x_1), x_3 = T_1(x_2), \quad \ldots$$

For any set A, let $\delta(A)$ denotes the diameter of A.

Assume, $O(x_0, n) = \{x_0, x_1, \ldots, x_n\}$

and $\quad O(x_0, \infty) = \{x_0, x_1, \ldots\}$

Let $K \leqslant i < j < n$, $x = x_i$, $y = x_j$

where i is an even integer and j is an odd integer,

$z = x_{i-k}$ and $p+q=k$.

Mow,

$$d(x_{i+1}, x_{j+1}) = d(T_1(x_i), T_2(x_j))$$

$$\leqslant h\, max\, \{ d(x_i, x_j), d(x_i, x_{i-k}), d(x_j, x_{i-k}),$$

$$d(x_i, x_{i+1}), d(x_j, x_{j+1}), d(x_i, x_{j+1}), d(x_j, x_{i+1}) \}$$

$$\leqslant h\, \delta\, [0(x_{i-k}, j+1+k-i)]$$

$$\leqslant h\, \delta\, [0(x_{i-k}, n)] \ldots \quad (2)$$

Note that $\delta\, [0(x_{i-k}, 1)] \leqslant \delta\, [0(x_{1-k}, 2)] \leqslant \ldots$

so that $\delta\, [0(x_{i-k}, \infty)] = Sup_n \delta\, [0(x_{i-k}, n)]$.

From (2) it follows that,

$$\delta\, [0(x_{i-k} . n)] = d(x_e, x_t)$$

for some choice of integer c, t satisfying,

$i\text{-}k \leqslant c \leqslant t,\ c < t \leqslant n$.

Let n be any positive integer. Then there cxists intcgers c and t with

$i\text{-}k \leqslant c \leqslant i,\ c < t \leqslant n$ such that

$$\delta\, [0(x_{i-k}, n)] = d(x_c, x_t)$$

If $t \leqslant i+1$, then

$$d(x_c, x_t) \leqslant M = \max\{d(x_c, x_u) : i-k \leqslant c < u \leqslant i+1\}$$

If $t > i$, then,

$$d(x_c, x_t) \leqslant d(x_c, x_{i+1}) + d(x_{i+1}, x_t)$$

$$\leqslant M + h\,\delta[(0(x_{i-k}, n)]$$

$$\leqslant M + hd(x_c, x_t) \text{ and } d(x_c, x_t) \leqslant \frac{M}{1-h}$$

Therefore, $\delta[0\, x_{i-k}, \infty)] \leqslant \frac{M}{1-h}$.

Let $m > n > k$, From (2) with

$i+1=n$, $j+1=m$,

$$d(x_m, x_n) \leqslant h\,\delta[0(x_{n-k-1}, m+k-n+1)]$$

Then there exists a cho:ce of integers c_1, t_1 satisfying $0 \leqslant c_1 \leqslant k$, $c_1 < t_1 \leqslant m+k-n+1$ snch that

$$\delta[0(x_{n-k-1}, m+k-n+1)] = d(x_{n-k-1+c_1}, x_{n-k-1+t_1})$$

$$= d[T_1(x_{n-k-2+c_1}), T_2(x_{n-k-2+t_1})]$$

$$\leqslant h\,\delta[0(x_{u-2k-2+c_1}, t+k+1-c_1)]$$

$$\leqslant h\,\delta[0(x_{n-2k-2}, m-n+2k+2).$$

So that,

$$d(x_m, x_n) \leqslant h^2\delta[0(x_{n-2k-2}, m-n+2k+2)]$$

In general, with r satisfying

$$n = r(k+1) + s,\ 0 \leqslant s < k+1 .$$

$$d(x_m, x_n) \leqslant h^r \delta[\,0(x_s, m-n+r(k+1))\,]$$

$$\leqslant h^r \delta[\,0(x_0, m)\,]$$

$$\leqslant h^r \frac{M}{1-h} \to 0 \text{ as } r \to \infty .$$

Therefore $\{x_n\}$ is a Cauchy sequence, hence convergent to a limit say x'.

Now for even n, consider $x = x_n$, $y = x'$, $z = x_{n-k}$. $p+q = k$.
Then by (1) we have,

$$d(x_{n+1}, T_2(x')) = d(T_1(x_n), T_2(x'))$$

$$\leqslant h\, mnx\, \{ d(x_n, x'), d(x_n, x_{n-k}),$$

$$d(x', x_{n-k}), d(x_n, x_{n+1}), d(x', T_2(x')),$$

$$d(x_n, T_2(x')); d(x', x_{n+1}) \}$$

Taking the limit as $n \to \infty$, we obtain

$$d(x', T_2(x')) \leqslant hd(x', T_2(x')) \Rightarrow T_2(x') = x'$$

i. e., x' is the fixed point of T_2.

Simiarly by taking n odd and

$$x = x',\ y = x_n,\ z = x_{n-k}$$

We get from (1) ,

$$d(T_1(x'), x_{n+1}) = d(T_1(x'), T_2(x_n))$$

$$\leqslant h\{d(x', x_n)\, d(x', x_{n-k}),\ d(x_n, x_{n-k}),$$

$$d(x', T_1(x')),\ d(x_n, x_{n+1}),\ d(x', x_{n+1}),$$

$$d(x_n, T_1(x))\}.$$

Tnen taking the limit as $n \to \infty$, we obtain

$$d(T_1(x'), x')) \leqslant h\, d(x', T_1(x') \Rightarrow T_1(x') = x',$$

Showing that x' is the fixed point of T_1 ,

Thus x' is a common fixed point of T_1 and T_2 .

Now we shall show that there is no other common fixed point y' of T_1 and T_2 . For is y' were such a point then , $T_1(y') = T_2(y') = y', T_1(x') = T_2(x') = x'$. considering $x = z = x'. y = y'$, we get

$$d(x', y') = d(T_1(x'), T_2(y')) \leqslant hd(x', y')$$

showing that $x' = y'$

Remark .

If we put $T_1 = T_2$ in theorem 1, we get the following theorem of Rhoades :

Let f be a self mapping of a eomplete metric space X which satisfying for each $x, y, z \in X$, $0 \leqslant h \leqslant 1$ and a positive integer k,

$$d(f(x), f(y)) \leqslant h \max \{ d(x, y), d(x, z), d(y, z), d(x, f(x)),$$

$$d(y, f(y)), d(x, f(y)), d(y, f(x)),$$

$$[d(x, f^k(z)) + d(y, f^k(z))], [d(x, f^k(z)) + d(f(x), f^k(z))],$$

$$\frac{[d(y, f^k(z)) + d(f(y), f^k(z))]}{2}, [d(x, f^k(z)) + d(f(y), f^k(z))],$$

$$\frac{[d(y, f^k(z)) + d(f(x), f^k(z))]}{2} \}.$$

Theorem 2.

Consider for any positive integers k and m, the self mapping T of a complete metric space X satisfies for $x, y, z_1, z_2 \in X$, $x \neq y$,

$$d(T(x), T(y)) < \max \{ d(x, y), d(x, T(x)), d(y, T(y)),$$

$$\frac{[d(x, T(y)) + d(y, T(x))]}{2}, [d(x, T^k(z_1)) + d(y, T^k(z_1))],$$

$$[d(x, T^m(z_2)) + d(y, T^m(z_2))], \frac{[d(y, T^k(z_1)) + d(T(y), T^k(z_1))]}{2},$$

$$\frac{[d(y, T^m(z_2)) + d(T(y), T^m(z_2))]}{2},$$

$$[d(x, T^k(z_1)) + d(T(y), T^k(z_1))],$$

$$[d(x, T^m(z_2)) + d(T(y), T^m(z_2))],$$

$$[d(x, T^k(z_1)) + d(T(x), T^k(z_1))],$$

$$[d(x, T^m(z_2)) + d(T(x), T^m(z_2))] \ldots \quad (1)$$

If $T^n(x_0)$ has a cluster point p for some $x_0 \in X$ and if T is continuous on cl $0(x_0)$, the closure of the orbit of x_0, then p is the uniqne fixed point of T and $T^n(x_0) \to p$.

Proof . Let $x_0 \in X$ and consider the sequence $\{x_n\}$,

$\{ x_0, x_1 = T(x_0), \ldots, x_{n+1} = T(x_n), \ldots \}$.

For each $k,m \leqslant i$, take, $x = x_i$, $y = x_{i-1}$, $z_1 = x_{i-k}$, $z_2 = x_{i-m}$.

Then we have from (1),

$$d(x_{i+1}, x_i) < max \{ d(x_i, x_{i-1}), d(x_i, x_{i+1}), d(x_i, x_{i+1}), d(x_{i-1}, x_i), \frac{[d(x_i, x_i) + d(x_{i-1}, x_{i+1})]}{2}, [d(x_i, x_i) + d(x_{i-1}, x_i)],$$

$$[d(x_i, x_i) + d(x_{i-1}, x_i)], \frac{[d(x_{i-1}, x_i) + d(x_i, x_i)]}{2},$$

$$\frac{[d x_{i-1}, x_i) + d(x_i, x_i)]}{2}, [d(x_i, x_i) + d(x_i, x_i)],$$

$$[d(x_i, x_i) + d(x_i, x_i)], [d(x_i, x_i) + d(x_{i+1}, x_i)],$$

$$[d(x_i, x_i) + d(x_{i+1}, x_i)]$$

$$i.e., d(x_{i+1}, x_i) \leqslant max \{ d(x_i, x_{i-1}), \frac{(x_{i-1}, x_{i+1})}{2} \} \quad \ldots (2)$$

We have for $x_{i+1} \neq x_i$, since each $i \geqslant k, m$; otherwise $\{ T^n(x_0) \}$ automatic converges to a fixed point .

Suppose the maximum of the right hand side of (2) is

$$\frac{d(x_{i-1}, x_{i+1})}{2} .$$

Then we have ,

$$d(x_{i-1}, x_i < \frac{d(x_{i-1}, x_{i+1})}{2}$$

$$\leqslant [\frac{d(x_{i-1}, x_i) + d(x_i, x_{i+1})}{2}]$$

which implies $d(x_{i-1}, x_i) < d(x_i, x_{i+1})$ ... (3)

Also $d(x_{i+1}, x_i) < \frac{d(x_{i-1}, x_{i+1})}{2}$ which implises

$$d(x_{i+1}, x_i) < \frac{d(x_{i-1}, x_i)}{2} + \frac{d(x_i, x_{i+1})}{2}$$

i e., $d(x_{i+1}, x_i) < \frac{d(x_{i-1}, x_i)}{2} < d(x_{i-1}, x_i)$ ... (4)

Combining (3) and (4) we have the contradiction,

$$d(x_{i+1}, x_i) < d(x_{i+1}, x_i) .$$

Therefore from (2) , $d(x_{i+1}, x_i) < d(x_i, x_{i-1})$ [4, Lemma 2]

This shows that $T(p) = p$.

Suppose p and w are fixed points of T, $p \neq w$.

Then from (1) we obtain the contradiction, $d(p, w) < d(p, w)$.

Since $T^n(x_0) \to p$ for some subsequence $\{n_i\}$ and T is continuous on $cl\ 0(x_0)$,

$$d(T^{n_i}(x_0), T^{n_i+1}(x_0)) = d(x_{n_i}, x_{n_{i+1}}) \to d(p, T(p))$$

$$= 0 \Rightarrow T(p) = p.$$

Fix $\varepsilon > 0$. Then there exists an integer j such that $i \geqslant j$ implies

$d(x_{n_i}, x_{n_{i+1}}) < \varepsilon$ and $d(x_{n_i}, p) < \varepsilon$

If we choose $x = x_{n-1}$, $y = p$, $z_1 = x_{n-k-1}$ and $z_2 = x_{n-m-1}$,

then for each $n > n_{j+k}$, $n > n_{j+m}$, we have from (1) and (4),

$$d(x_n, p) < \max\{d(x_{n-1}, p), d(x_{n-1}, x)\}$$

$$< \max\{d(x_{n-2}, p), d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}$$

$$= \max\{d(x_{n-2}, d(x_{n-2}, x_{n-1})\} \text{ from (4)}$$

$$< \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$< \max\{d(x_{n_j}, x_{n_{j+1}}), d(x_{n_j}, p)\} < \varepsilon$$

i. e., $T^n(x_0) \to p$.

Theorem 3.

Let T_1 and T_2 be two mappings on a complete metric space X on itself such that $T_1^p\ T_2^q = T_2^q\ T_1^p$; $p, q \geqslant 1$, then for every $x, y, z_1, z_2 \in X$ and $0 < h < 1$,

$$d(T_1(x), T_2(y)) \leqslant h \max \{ d(x, y), d(x, T_1(x)), d(y, T_2(x)),$$

$$\frac{[d(x, T_2(y)) + d(y, T_1(x))]}{2},$$

$$[d(x, T_1^p T_2^q (z_1)) + d(y, T_1^p T_2^q (z_1))],$$

$$[d(x, T_2^p T_2^q (z_2)) + d(y, T_1^p T_2^q (z_2))],$$

$$[d(x, T_1^p T_2^q (z_1)) + d(T_1(x), T_1^p T_2^q (z_1))],$$

$$[d(x, T_1^p T_2^q (z_2)) + d(T_1(x), T_1^p T_2^q (z_2))],$$

$$[d(y, T_1^p T_1^q (z_1)) + d(T_2(y), T_1^p T_2^q (z_1))],$$

$$[d(y, T_1^p T_2^q (z_2)) + d(T_2(y), T_1^p T_2^q (z_2))],$$

$$[d(x, T_1^p T_2^q (z_1)) + d(T_2(y), T_1^p T_2^q (z_1))],$$

$$[d(x, T_1^p T_2^q (z_2)) + d(T_2(y), T_1^p T_2^q (z_2))] \quad \ldots (1)$$

Then T_1 and T_2 have a unique common fixed point.

Proof .

Let $x_0 \in X$ and constrtuct a sequence $\{x_n\}$ as follows :

x_0, $T_1(x_0) = x_1$, $T_2(x_1) = x_2$, $T_1(x_2) = x_3$, $T_2(x_3) = x_4$, and so on .

Consider $x = x_i$ (i - even) ,

$y = x_{i-1}$, $z_1 = x_{i-p-q}$, $z_2 = x_{i-p-q-1}$. Then from (1) we have ,

$$d(T_1(x), T_2(y)) = d(T_1(x_i), T_2(x_{i-1})) = d(x_i, x_{i+1})$$

$$\leqslant h \max \{ d(x_i, x_{i-1}), d(x_i, x_{i+1}), d(x_{i-1}, x_i),$$

$$\frac{[d(x_i, x_i) + d(x_{i-1}, x_{i+1})]}{2}, [d(x, , x_i) + d(x_{i-1}, x_i)],$$

$$[d(x_i, x_i) + d(x_{i-1}, x_i)], [d(x_i, x_i) + d(x_{i-1}, x_i)],$$

$$[d(x_i, x_i) + d(x_{i-1}, x_i)], [d(x_{i-1}, x_i) + d(x_i, x_i)],$$

$$[d x_{i-1}, x_i) + d(x_i, x_i)], [d(x_i, x_i) + d(x_i, x_i)],$$

$$[d(x_i, x_i) + d(x_i, x_i)] \},$$

$$\leqslant h \max \{ d(x_i, x_{i-1}), \quad d(x_i, x_{i+1}), d \frac{(x_{i-1}, x_{i+1})}{2} \}$$

$$< h \max \{ d(x_i, x_{i-1}), \frac{d(x_{i-1}, x_{i+1})}{2} \}.$$

This implies that $d(x_i, x_{i+1}) \leqslant h\, d(x_i, x_{i-1})$

i. e. , $d(T_1(x), T_2(y)) \leqslant h\, d(x, y)$.

This shows that T_1 and T_2 have a common fixed point [6] .

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