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ON CERTAIN TRIPLE INTEGRAL RELATIONS

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ABSTRACT

The aim of this paper is to establish three triple integral relations involving elementary functions. A number of triple integrals can be deduced by proper specialization of the unknown functions g and f occurring in these relations. For the sakc of illustration, one of our integral relations is applied to evaluate a general triple integral involving Srivastava and Panda's multivariable H-function.

1. INTRODUCTION

Many authors have worked on the problem of obtaining integral relatious Involving higher classes of special functions of one and more variables (see [3, pp. 72-74 ; pp. 156-161] for details). In this paper we derive three new integral ralations associated with some elementary functions and illustrate how they can be applied to derive triple integrals which may be of interest.

2. INTEGRAL RELATIONS

$$\int_0^\infty\int_0^\infty\int_0^\infty (x^2+y^2)^{-1/2} \, exp\left[i (x^2+y^2+z^2) \left(\frac{x^2-y^2}{x^2+y^2} \right) \right]$$

$$cos\left[2n \, tan^{-1} (y/x) \right] f(x^2+y^2+z^2)$$

$$\cdot\ g\ \{\ \tan^{-1}\ (\frac{x^2+y^2}{2})\ \}\ dx\ dy\ dz$$

$$=\frac{\pi i^n}{2}\int_0^\infty\int_0^\infty J_n\,(\,u^2+v^2\,)\,f\,(\,u^2+v^2\,)\ g\ \{\ \tan^{-1}\ (\frac{v}{u})\ du\ dv\,, \qquad \ldots (2.1)$$

where n is any integer, positive or negative, and the functions f and g are so constrained that the various integrals involved in (2.1) exist.

$$\int_0^\infty\int_0^\infty\int_0^\infty (xy)^{\nu+1/2}\,(x^2+y^2)^{-(\nu+1)}\ exp\ [\,i\,(\,x^2+y^2+z^2\,)\,(\frac{x^2-y^2}{x^2+y^2})\ \cos\Psi\,]$$

$$\cdot\, J_{\nu-1/2}\ [\,\frac{2xy}{(x^2+y^2)}\ (\,x^2+y^2+z^2\,)\ \sin\ \Psi\ ]\ C_n^\nu\ (\frac{x^2-y^2}{x^2+y^2})\,f\,(\,x^2+y^2+z^2\,)$$

$$g\ [\ \tan^{-1}\,(\frac{x^2+y^2}{z})\ ]\ dx\ dy\ dz$$

$$=\ 2^{-(\nu+1)}\ \sqrt{\pi}\, i^n\,(\,\sin\ \Psi\ )^{\nu-1/2}\quad C_n^\nu\ (\ \cos\ \Psi\ )$$

$$\cdot\int_0^\infty\int_0^\infty (\,u^2+v^2\,)^{-1/2}\ J_{\nu+n}\,(\,u^2+v^2\,)\,f\,(\,u^2+v^2\,)\ g\ [\ \tan^{-1}\ v/u\ ]\ du\ dv \qquad \ldots (2.2)$$

provided that $Re\,(\nu) > -1/2$, $n = 0,1.2, \ldots,$ and f and g are so constrained that the various integrals involved in (2.2) exist.

$$\int_0^\infty\int_0^\infty\int_0^\infty (\,xy\,)^{2\nu}\,(\,x^2+y^2\,)^{-(2\nu+1/2)}\ W^{-\nu}\ Z_\nu\,(\,W\,)$$

$$C_n^{\nu}\left(\frac{x^2-y^2}{x^2+y^2}\right) f(x^2+y^2+z^2)\, g\left[\tan^{-1}\left(\frac{x^2+y^2}{z}\right)\right] dx\, dy\, dz$$

$$=\frac{2^{3\nu}\pi\,\Gamma(n+2\nu)\, Z_{\nu+n}(t)\, t^{-\nu}}{n!\,\Gamma(\nu)}$$

$$\cdot \int_0^{\infty}\int_0^{\infty} (u^2+v^2)^{-\nu} \quad J_{\nu+n}(u^2+v^2)\, f(u^2+v^2)\, g\,[\tan^{-1} v/u]\, du\, dv \qquad \ldots (2.3)$$

valid under the same conditions as those stated for (2.2) above .

In (2.1) , (2.2) and (2.3) $J_\nu(x)$ is the Bessel function of the first kind, $C_n^{\nu}(x)$ is the Gegenbauer polynomial, and $Z_\nu(x)$ stands for any Bessel function of the first, second, or third kind . Also

$$W = [\,(x^2+y^2+z^2)^2 + t^2 - 2t\left(\frac{x^2-y^2}{x^2+y^2}\right)(x^2+y^2+z^2)\,]^{-1/2} \qquad \ldots (2.4)$$

Proof of (2.1) : We have [5,p. 409, Ex. 3]

$$J_n(z) = \frac{1}{\pi i^n}\int_0^{\pi} exp\,(\,iz\cos\theta\,)\,\cos n\theta\, d\theta \qquad \ldots (2.5)$$

In order to derive the iutegral relation (2.1), we replace z by r^2,θ by 2θ in (2.5), multiply both sides by $rf(r^2)\, g(\phi)\, dr\, d\phi$, and then integrate the resulting equation with respect to r and ϕ over the intervals $(0,\infty)$ and $(0,\pi/2)$, respectively . We thus get

$$\frac{\pi i^n}{2}\int_0^{\infty}\int_0^{\pi/2} \{J_n(r^2)\, rf(r^2)\; g(\phi)\, dr\}\; d\phi$$

$$= \int_0^{\infty} \int_0^{\pi/2} \int_0^{\pi/2} exp\,(ir^2 \cos 2\theta) \cos 2n\theta \, \frac{r \sin \phi}{r \sin \phi} \, rf(r^2)\, g(\phi)\, dr\, d\phi\, d\theta \qquad \ldots (2.6)$$

If we make the substitutions $x = r \sin \phi \cos \theta$, $y = r \sin \phi \sin \theta$, and $z = r \cos \phi$ on right-hand side of (2.6), and set $u = r \cos \phi$, $v = r \sin \phi$ in left hand side, we are easily led to the integral relation (2.1).

To prove the integral relations (2.3) and (2.4), we start with the known integrals [1, p. 57, Eq. (12) ; p. 48, Eq. (14)] and proceed on the lines similar to those mentioned in the proof of (2.1).

3. USEFUL DEDUCTIONS

The function g appearing in our integrals (2.1) (2.2) and (2 3) may be chosen appropriately to derive various triple integrals .

For example, if in (2.1), we get

$$g(t) = \cos 2\mu t)\,(\sin t)^{\nu} \qquad \ldots (3.1)$$

and simplify the right-hand side of the resulting equation by means of a known integral relation [3, p. 72, Eq. (5. 4. 3)], we arrive at the following result :

$$\int_0^{\infty} \int_0^{\infty} \int_0^{\infty} (x^2+y^2)^{(\nu-1)/2} \, (x^2+y^2+z^2)^{-\nu/2} \, exp\, [\, i\,(x^2+y^2+z^2)$$

$$(\frac{x^2-y^2}{x^2+y^2})\,]\, \cos\, [\,2n\,(\tan^{-1} y/x\,)\,]\, \cos\, [\,2\mu\,\{\tan^{-1}\,(\frac{x^2+y^2}{z}\,\}\,]$$

$$f(x^2+y^2+z^2)\, dx\, dy\, dz$$

$$= \frac{\sqrt{\pi}\, i^n\, \Gamma(\tfrac{1}{2} \pm \mu)\, \Gamma(\frac{1+\nu}{2})\, \Gamma(1+\nu/2)}{8\, \Gamma(1+\nu/2 \pm \mu)} \int_0^\infty f(t)\, J_n(t)\, dt \qquad \ldots (3.2)$$

Re (ν) $>$ 0, n is an integer, positive or negative, and f is so chosen that the integrals on both sides of (3.2) exist .

Now in (3.2), we set

$$f(t) = t^{\lambda-1} H[z_1 t^{\rho_1}, \ldots, z_r t^{\rho_r}]$$

where

$$H[z_1, \ldots, z_r) \equiv H^{O,O:M_1,N_1;\ldots;M_r,N_r}_{P,Q:P_1,Q_1;\ldots;P_r,Q_r}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (a_j;\ \alpha'_j, \ldots, \alpha^{(r)}_j)_{1,P} : \\ (b_j;\ \beta'_j, \ldots, \beta^{(r)}_j)_{1,Q} : \end{matrix} \right.$$

$$\left. \begin{matrix} (c'_j, \gamma'_j)_{1,P_1}; \ldots; (c^{(r)}_j, \gamma^{(r)}_j)_{1,P_r} \\ (d'_j, \delta'_j)_{1,Q_1}; \ldots; (d^{(r)}_j, \delta^{(r)}_j)_{1,Q_r} \end{matrix} \right]$$

is a special case of the multivariable H-function due to Srivastava and Panda [4] (see also [3, p. 251, Eq. (C. 1)]) .

Evaluating the resulting integral with the help of a known integral [2, p. 272, Eq. (3. 6)] , we arrive at the following interesting triple integral which is believed to be new :

$$\int_0^\infty\int_0^\infty\int_0^\infty (x^2+y^2+z^2)^{\lambda-(\nu+2)/2} (x^2+y^2)^{(\nu-1)/2} exp [i (x^2+y^2+z^2)$$

$$(\frac{x^2-y^2}{x^2+y^2})] \cos [2n (tan^{-1} y/x)] \cos \{ 2 \mu (tnn^{-1} \frac{x^2+y^2}{z}) \}$$

$$H [Z_1 (x^2+y^2+z^2)^{\rho_1}, \ldots, Z_r (x^2+y^2+z^2)^{\rho_r}] \, dx \, dy \, dz$$

$$= \frac{2^{\lambda-4} \sqrt{\pi}\, i^n \, \Gamma(\tfrac{1}{2} \pm \mu\, \Gamma(\frac{1+\nu}{2}) \Gamma(1+\frac{\nu}{2})}{\Gamma(+\frac{\nu}{2} \pm \mu)}$$

$$H^{O,1:M_1,N_1;\ \ldots;M_r,N_r}_{P+2,Q:P_1,Q_1;\ldots;P_r,Q_r}\left[\begin{matrix} Z_1 2^{\rho_1} \\ \vdots \\ Z_r 2^{\rho_r}\end{matrix}\middle|\begin{matrix}(1\mp\frac{n}{2}-\frac{\lambda}{2};\frac{\rho_1}{2},\ldots,\frac{\rho_r}{2})\\ \\ \end{matrix}\right.$$

$$\left.\begin{matrix}(a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P};(c_j',\gamma_j')_{1,P_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,P_r}\\ (b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q};d_j',\delta_j')_{1,Q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,Q_r}\end{matrix}\right]$$

... (3 3)

provided that $Re(\nu) > 0$, $\rho_i > 0$ ($i = 1, \ldots, r$), n is an integer, positive or negative,

$$Re(\lambda) + \sum_{i=1}^{r} [\rho_i \min_{1\leqslant j\leqslant m_i} Re(d_j^{(i)}/\delta_j^{(i)})] + n > 0,$$

$$Re(\lambda) + \sum_{i=1}^{r} \left[\rho_i \max_{1 \leqslant j \leqslant n_i} Re\left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right] - \frac{3}{2} < 0$$

$$\Lambda_i = - \sum_{j=1}^{P} \alpha_j^{(i)} - \sum_{j=1}^{Q} \beta_j^{(i)} + \sum_{j=1}^{N_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{P_i} \gamma_j^{(i)}$$

$$+ \sum_{j=1}^{M_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{Q_i} \delta_j^{(i)} > 0$$

and

$$| \arg(Z_i) | < \tfrac{1}{2} \Lambda_i \pi, \; (i = 1, \ldots, r).$$

The triple integral (3.3) is quite general in character due to general nature of the multivariable H-function involved therein. Thus, by appropriately reducing this multivariable H-function in terms of simpler special functions, one can easily obtain a considerably large number of triple integrals of interest to mathematical analysts and applied mathematicians. The details are omitted.

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