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RAYLEIGH-TAYLOR INSTABILITY OF A FLUID-PARTICLE MIXTURE IN POROUS MEDIUM

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ABSTRACT

The Rayleigh - Taylor instability of a fluid - particle mixture in porous medium is considered. For the case of two uniform fluids separated by a horizontal boundary, the stable configuration (lighter fluid overlying the heavier one) remains stable but for unstable configuration (heavier fluid overlying the lighter one), the system is unstable and the growth rates may increase or decay with the increase in particle number density. For the case of exponentially varying density, there is instability for the unstable stratification and the growth rates are found to be both increasing or decreasing with the increase in suspended particle number density and medium permeability.

1. INTRODUCTION

The instability derived from the character of the equilibrium of an incompressible heavy fluid of variable density (*i. e.* of a heterogeneous fluid) is termed the Rayleigh-Taylor instability. Mention may be made of two important special cases : (a) two fluids of different densities superposed one over the other ; (b) a fluid with a continuous density stratification . Chandrasekhar [1] has given a detailed account of the Rayleigh-Taylor instability in non-porous medium . When a fluid permeates a porous material, the gross effect is represented by by the Darcy's law . As a result of this macroscopic law, the viscous form in the equations of fluid motion is replaced by the resistance term (μ/k_1) v, where μ is the viscosity of the fluid, k_1 is the permeability of the medium and v is the Darcian (filter) velocity of the fluid . Wooding [4] has considered the Rayleigh instability of a thermal boundary layer in flow through a porous medium .

In geophysical situations . more often than not, the fluid is not pure but may instead be permeated with suspended (or dust) particles, Scanlon and Segel [2] have considered the effect of suspended particles on the onset of Bénard convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure gas was supplemented by that of the particles , The thermal instability of fluids in a porous medium in the presence of suspended particles has been studied by Sharma and Sharma [3] . The effects of suspended particles and medium permeability were found to destabilize the layer .

Keeping in mind the geophysical situations and motivated by interest in fluid-particle mixtures, a study has been made of the Rayleigh-Taylor instability of a fluid-particle mixture in porous medium in the present note ,

2. Formulation of the Problem and Perturbation Equations

Consider an incompressible fluid-particle layer consisting of a fluid of density ρ , permeated with suspended particles of density mN, arranged in horizontal strata in porous medium . The system is acted on by gravity force g (0,0,$-g$) . Let p ρ and v (u,v,w) denote respectively the pressure, density and velocity of the pure fluid ; **u** (1,r,s), m and $N(\overline{x}, t)$ denote the velocity, mass and number density of the particles respectivety . $K = 6\pi\rho\nu\eta$, where η is particle radius, is constant, ε is porosity and $\overline{x} = (x, y, z)$, In the equations of motion for the fluid, the presence of particles adds an extra force term, proportional to the velocity difference between particles and fluid . The equations of motion and continuity for the fluid are

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\varepsilon}\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}\right] = -\nabla p + \rho \mathbf{g} - \frac{\mu}{k_1}\mathbf{v} + \frac{KN}{\varepsilon}\left(\mathbf{u} - \mathbf{v}\right), \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2}$$

Following a fluid particle in motion, we have

$$\varepsilon\frac{\partial \rho}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\rho = 0 . \tag{3}$$

Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion for the particles . The buoyancy force on the paticles is neglected, The interparticle reactions are also not considered because we assume that the distances between particles are quite large compared with their diameter . Under the above assumptions, the equations of motion and continuity for the particles are

$$mN\left[\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\varepsilon}\left(\mathbf{u}\cdot\nabla\right)\right] = KN\left(\mathbf{v} - \mathbf{u}\right), \tag{4}$$

$$\varepsilon\frac{\partial N}{\partial t}+\nabla\cdot\left(N\mathbf{u}\right)=0. \tag{5}$$

The initial state of the system is taken to be a quiescent layer (no settling) with a uniform particle distribution N_0 *i. e.* $\mathbf{v}=(0,0,0)$, $\mathbf{u}=(0,0,0)$ and $N=N_0$, is a constant , The character of the equilibrium of this initial static state can be determined, as usual, by supposing that the system is slightly disturbed and then by following its further evolution .

Lel $\delta\rho$, δp, $\mathbf{v}(u, v, w)$ and $\mathbf{u}(l, r, s)$ denote respectively the perturbations in density ρ . pressure p. velocity of fluid and velocity of particles . Then the linearized perturbed forms of eqs . (1) - (5) become

$$\frac{\rho}{\varepsilon}\frac{\partial\mathbf{v}}{\partial t}=-\nabla\delta p+\mathbf{g}\delta\rho-\frac{\rho\nu}{k_1}\mathbf{v}+\frac{KN}{\varepsilon}\left(\mathbf{v}-\mathbf{u}\right), \tag{6}$$

$$\nabla\cdot\mathbf{v}=0,$$

$$mN\frac{\partial\mathbf{u}}{\partial t}=KN\left(\mathbf{v}-\mathbf{u}\right), \tag{8}$$

$$\varepsilon\frac{\partial}{\partial t}=-\omega\frac{d\rho}{dz}, \tag{9}$$

wnere $\nu\ (=\mu/\rho)$ stands for kinematic viscosity of the fluid .

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y and t is of the form

$$exp\left(ik_x x+ik_y y+nt\right), \tag{10}$$

where n is a complex constant and k_x, k_y ($k^2 = k_x^2 + k_y^2$) are wave numbers along x and y directions respectively.

Eliminating **u** between eqs. (6) and (8) and using (10), eqs. (6)-(9) give

$$\left(n'+\frac{\nu}{k_1}\right)\rho u = -ik_x \delta p , \tag{11}$$

$$\left(n'+\frac{\nu}{k_1}\right)\rho v = -ik_y \delta p , \tag{12}$$

$$\left(n'+\frac{\nu}{k_1}\right)\rho w = -D\delta p - g\delta\rho, \tag{13}$$

$$ik_x u + ik_y v + D\omega = 0 , \tag{14}$$

$$\varepsilon n\delta\rho = -\omega D\rho , \tag{15}$$

where

$$n' = \frac{n}{\varepsilon}\left(1+\frac{uNK/\rho}{mu+K}\right) \text{ and } D=d/dz .$$

Eliminating δp between eqs. (11)-(13) and using eqs. (14) and (15), we obtain

$$n' [D(\rho D\omega) - k^2\rho\omega] + 1/k_1 [D(\rho\nu D\omega) - k^2\rho\nu\omega]$$

$$= -\frac{gk^2}{\varepsilon n}(D\rho)\,\omega . \tag{16}$$

3. Two uniform fluids separated by a horizontal boundary

Here we consider the case when two superposed fluids of uniform densities ρ_1 and ρ_2 and uniform viscosities μ_1 and μ_2 are separated by a horizontal boundary at $z=0$. The subscripts 1 and 2 distinguish the lower and the upper fluids respectively. Then, in each region of constant ρ and constant μ, eq. (16) reduces to

$$(D^2-k^2)\,w = 0\,. \tag{17}$$

The general solution of eq. (17) is

$$\omega = A\,e^{+kz} + B\,e^{-kz}\,, \tag{18}$$

where A and B are arbitrary constants.

The boundary conditions to be satisfied in the present problem are :

(i) The velocity ω should vanish when $z \to +\infty$ (for the upper fluid) and $z \to -\infty$ (for the lower fluid).

(ii) ω (z) is continuous at $z=0$.

(iii) The pressure should be continuous across the interface.

Applying the boundary conditions (i) and (ii). we have

$$\omega_1 = A\,e^{+kz}\ (z < 0), \tag{19}$$

$$\omega_2 = A\,e^{-kz}\ (z > 0), \tag{20}$$

the same constant A has been chosen to ensure the continuity of w at $z = 0$.

The continuity of pressure means that

$$n'\Delta_0\left(\rho D\omega\right)+\frac{1}{k_1}\,\Delta_0\left(\rho\nu D\omega\right)+\frac{gk^2}{n\varepsilon}\,\Delta_0(\rho)\,\omega_0=0\,. \tag{21}$$

Applying the condition (21) to the solutions (19) and (20), we obtain

$$n^3+\left[\frac{K}{m}+\frac{2KN}{\rho_1+\rho_2}+\frac{\varepsilon}{k_1}\left(\alpha_1\nu_1+\alpha_2\nu_2\right)\right]n^2$$

$$+\left[\frac{K}{m}\,\frac{\varepsilon}{k_2}\left(\alpha_1\nu_1+\alpha_2\nu_2\right)+gk\left(\alpha_1-\alpha_2\right)\right]n+\frac{gkK}{m}\left(\alpha_1-\alpha_2\right)=0\,, \tag{22}$$

where $\alpha_{1,2}=\dfrac{\rho_{1,2}}{\rho_1+\rho_2}$ and $\nu_{1,2}=\dfrac{\mu_{1,2}}{\rho_{1,2}}$.

(a) *Stable Case* $(\rho_1 > \rho_2)$

For potentially stable case, $\alpha_1 > \alpha_2$ and eq. (22) does not involve any change of sign and so does not allow any positive root. Therefore the system is stable.

(b) Unstable Case $(\rho_2 > \rho_1)$

For potentially unstable case, eq. (22) has one positive root and so the system is unstable. Let n_0 denote the positive root of eq. (22). Then eq. (22) yields

$$n_0^3+\left[\frac{K}{m}+\frac{2KN}{\rho_1+\rho_2}+\frac{\varepsilon}{k_1}\left(\alpha_1\nu_1+\alpha_2\nu_2\right)\right]n_0^2$$

$$+\left[\frac{K}{m}\frac{\varepsilon}{k_1}\left(\alpha_1\nu_1+\alpha_2\nu_2\right)+gk\left(\alpha_1-\alpha_2\right)\right]n_0+\frac{gkK}{m}\left(\alpha_1-\alpha_2\right)=0\,. \tag{23}$$

The presence of suspended particles, thus, does not affect the Rayleigh-Taylor instability in porous medium qualitatively. The deneminator on the right-hand side of eq. (23] may be positive or negative. Thus with the increase in particle number density for the unstable case, the growth rates may increase or decay.

4. The Case of Exponentially Varying Density

Here we consider the density stratification in the fluid of depth d as

$$\rho(z) = \rho_0 e^{\beta z}, \tag{24}$$

where ρ_0 and β are constants. Assume that [$\beta d << 1$, *i.e.*, the variation of density at two neighbouring points in the velocity field, which is much less than the average density, has a negligible effect on the inertia of the fluid,

The boundary conditions for the case of two free surfaces are

$$\omega = D^2\omega = 0 \text{ at } z = 0 \text{ and } z = d. \tag{25}$$

The proper solution of eq. (22) satisfying (25) is

$$\omega = W_0 \sin \frac{s\pi z}{d}, \tag{26}$$

where W_0 is a constant and s is an integer.

Substituting (26) in eq. (16) and neglecting the efsect of heterogeneity of the inertia, we get

$$n^3 + \left[\frac{k}{m}\left(1 + \frac{mN}{\beta}\right) + \frac{\varepsilon\nu}{k_1}\right]n^2 + \left[\frac{\varepsilon\nu}{k_1}\frac{K}{m} - \frac{g\beta k^2}{(\frac{s\pi}{d})^2 + k^2}\right]n$$

$$- \frac{g\beta k^2 K/m}{(\frac{s\pi}{d})^2 + k^2} = 0 . \tag{27}$$

For the stable stratification ($\beta<0$), eq. (27) does not have any positive root implying thereby that the system is stable . For the unstable stratification ($\beta>0$), the constant term in eq. (27) is negative . This meaus that eq. (27) possesses one positive root implying thereby that the system is unstable . Let n_0 denote the positive root of eq (27) . Then

$$n_0^3 + \left[\frac{K}{m}\left(1 + \frac{mN}{\rho}\right) + \frac{\varepsilon\nu}{k_1}\right] n_0^2 + \left[\frac{\varepsilon\nu}{k_1}\frac{K}{m} - \frac{g\beta k^2}{(\frac{s\pi}{d})^2 + k^2}\right] n_0$$

$$- \frac{g\beta k^2 K/m}{(\frac{s\pi}{d})^2 + k^2} = 0 , \tag{28}$$

To find the role of particle number density and medium permeability concerning the growth rate of unstable modes, we examine the natures of $d\, n_0/dN$ and dn_0/dk_1 analytically . Equation (28) yields

$$\frac{dn_0}{dN} = \frac{(K/\rho)\, n_0^2}{\frac{g\beta k^2}{(\frac{s\pi}{d})^2 + k^2} - \left[3n_0^2 + 2n_0\left\{\frac{K}{m}\left(1 + \frac{mN}{\rho}\right) + \frac{\varepsilon\nu}{k_1}\right\} + \frac{\varepsilon\nu}{k_1}\frac{K}{m}\right]}, \tag{29}$$

and

$$\frac{dn_0}{dk_1}=\frac{\frac{\varepsilon\nu}{k_1^2}\,n_0\left(n_0+\frac{K}{m}\right)}{\left[3n_0^2+2\,n_0\left\{\frac{K}{m}\left(1+\frac{\bar{m}N}{\rho}\right)+\frac{\nu}{k_1}\right\}+\frac{\varepsilon\nu}{k_1}\frac{K}{m}\right]-\frac{g\,\beta k^2}{\left(\frac{s\pi}{d}\right)^2+k^2}}. \tag{30}$$

The denominator on the right-hand side of eq. (29) may be positive or negative. Thus with the increase in particle number density for ihe unstable stratification, the growth rates may increase or decay. Similarly eq. (30) implies that with the increase in medium permeability, the growth rates may increase or decay for the unstable stratification.

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