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FIXED POINT THEOREM FOR MULTIVALUED MAPPINGS UNDER THE CARISTI - KIRK TYPE CONDITION

By

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ABSTRACT

In this paper we prove a fixed point theorem for a pair of multi-valued mappings under the Caristi-Kirk type condition. We also give the condition for the existence of such a fixed point.

1. INTRODUCTION

S. B. Nadler [2] proved a fixed point theorem for multivalued contraction mappings. We use the following notations and definitions as given by him in [2] for our work.

Definitions 1. *Let* (X, d) be a metric space, then

(i) $CB(X) = \{C : C$ is a nonempty closed and bounded subset of $X\}$

(ii) $N(\varepsilon, A) = \{x \in X \mid d(x, y) < \varepsilon$ for some $y \in A\}$,

$\varepsilon > 0$ and $A \in CB(X)$.

(iii) $H(A,B) = inf\{A \subset N(\varepsilon, B)$ and $B \subset N(\varepsilon, A)\}$,

$\epsilon > 0$ and $A, B \in CB(X)$.

The function H is a metric for $CB(X)$ called the Hausdorff metric for $CB(X)$.

Let (X, d_1) and (y, d_2) be two metric spaces. A function $F : X \to CB(y)$ is said to be a multivalued contraction mapping of X into y if

$$H(F((x), F(y)) \leqslant \alpha d_1(x, y) \; \forall \; x, y \in x,$$

$$0 \leqslant \alpha < 1$$

A point x is said to be a fixed point of a multivalued mapping F if $x \in F(x)$.

We shall also make use of the following lemmas, which are noted implicitly in Nadler [2]

Lemma I. *If $A, B \in CB(X)$ and $\bar{x} \in A$, then, for each positive number η, there exists a $y \in B$ such that*

$$d(\bar{x}, y) \leqslant H(A, B) + \eta .$$

Lemma 2. *Let $\{A_n\}$ be a sequence of sets in $CB(X)$, and suppose $\lim H(A_n, Ao) = 0$, where $Ao \in CB(X)$. Then, if $x_n \in A_n$, $n = 1, 2, 3. \ldots$, and $\lim x_n = x_0$, it follows that $x_0 \in A_0$*

Recently, Caristi ([1], Theorem 2. 1) presented a generalization of the contraction mapping principle which is now known as Caristi-Kirk fixed point theorem. The following proposition is observed by Felix Browder (See Caristi and Kirk [2]) as the modified version of the famous Caristi-Kirk theorem :

Proposition. *Let (X, d) be a complets metric space and $g : x \to X$ continuous. Suppose that there exists a lower semicontinuous function $\phi : X \to R^+$ such that*

$$d(x, gx) \leqslant \phi(x) - \phi(g(x)),\ x \in X .$$

Then, for any $x \in X$, *the sequence* $\{g^n(x)\}$ *converges to a fixed point at* g.

We prove

Theorem 1. *Let* (X, d) *be a complete metric space and* F_1, F_2 *be two continuous multivalued mappings on* X. *Let there be two mappings* ϕ *and* Ψ *on* $CB(x) \rightarrow [0, \infty)$ *such that*

(1.1) $H(F_1(x), F_2(y) \leqslant \phi(x) - \phi(F_1(x)) + \Psi(y) - \Psi(F_2\, y))$ *for all* $x, y \in X$.

(1.2) $\phi(A) \leqslant \phi(B)$ *and* $\Psi(A) \leqslant \Psi(B)$, A, B *in* $CB(x)$, *and* $A \subset B$.

Then F_1 *and* F_2 *have a common fixed point. Further, if any of the multivalued mappings is one-one, then this common fixed point is a unique common fixed point of the one-one mapping :*

Remark 1. We will use $\phi(x)$ instead of $\phi(\{x\})$ for the singleton set $\{x\}$, $x \in X$.

Proof. *Let* $x_0, y_0 \in X$ and $x_1 \in F_1(x_0)$. By Lemma 1, $\exists$ a $y_1 \in F_2(y_0)$ such that

$$d(x_1, y_1) < H(F_1(x_0), F_2(y_0)) + \tfrac{1}{2}$$

$$\leqslant \phi(x_0)) - \phi(F_1(x_0)) + \Psi(y_0) - \Psi(F_1(y_0)) + \tfrac{1}{2} \quad \text{by (1.1)}$$

Since $x_1 \in F_1(x_0)$. *i. e.*, $\{x_1\} \subset F_1(x_0)$ by (1.2), we have

$\phi(x_1) \leqslant \phi(F_1(x_0))$, Similarly, $\Psi(y_1) \leqslant \phi(F_2(y_0))$

Hence

$$d(x_1, y_1) < \phi(x_0) - \phi(x_1) + \Psi(y_0) - \Psi(y_1) + \tfrac{1}{2}$$

Since $y_1 \in F_2(y_0)$, $\exists$ an $x_2 \in F_1(x_1)$ such that

$$d(x_2, y_1) < H(F_1(x_1), F_2(y_0)) + \tfrac{1}{2}$$

$$\leqslant \phi(x_1) - \phi(F_1(x_1)) + \Psi(y_0) - \Psi(F_2(y_0)) + \tfrac{1}{2}$$

$$\leqslant \phi(x_1) - \phi(x_2) + \Psi(y_0) - \Psi(y_1) + \tfrac{1}{2}$$

Since $x_2 \in F_1(x_1)$, $\exists$ a $y_2 \in F_2(y_1)$ such that

$$d(x_2, y_2) < \phi(x_1) - \phi(x_2) + \Psi(y_1) - \Psi(y_2) + \frac{1}{2^2}$$

and, since $y_2 \in F_2(y_1)$. $\exists$ an $x_3 \in F_1(x_2)$ such that

$$d(x_3, y_3) < \phi(x_2) - \phi(x_3) + \Psi(y_1) - \Psi(y_2) + \frac{1}{2^2}$$

proceeding in this manner, we obtain $\{x_n\}$ and $\{y_n\}$ such that

$$x_n \in F_1(x_{n-1}),\ y_n \in F_2(y_{n-1}) \text{ and}$$

$$d(x_{n+1}, y_n) < \phi(x_n) - \phi(x_{n+1}) + \Psi(y_{n-1}) - \Psi(y_n) + \frac{1}{2^n}$$

So, for any positive integer n, we have

$$(1.3)\quad \sum_{r=1}^{n} d(x_r, y_r) < \sum_{r=1}^{n} \{\phi(x_{r-1}) - \phi(x_r) + \Psi(y_{r-1}) - \Psi(y_r)\} + \sum_{r=1}^{n} \frac{1}{2^r}$$

$$< \phi(x_0) - \phi(x_n) + \Psi(y_0) - \Psi(y_n) + 1$$

$$\leqslant \phi(x_0) + \Psi(y_0) + 1, \text{ since } \phi(x_n) \geqslant 0,$$

$$\Psi(y_n) \geqslant 0.$$

Similarly,

$$(1.4)\quad \sum_{r=1}^{n} d(x_r, x_{r+1}) \leqslant \sum_{r=1}^{n} [d(x_r, y_r) + d(x_{r+1}, y_r)]$$

$$< \phi(x_0) + \phi(x_1) + 2\Psi(y_0) + 2, \text{ and}$$

$\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ is convergent. So $d(x_n, x_{n+1}) \to 0$

as $n \to \infty$ Thus $\{x_n\}$ is a Cauchy sequence and converges to $z_1 \in X$. Since F_1 is continuous and $x_n \in F_1(x_{n-1})$ for $n = 1, 2, 3, \ldots$, by Lemma 2, z_1 is a fixed point of F_1.

Now

$$\sum_{r=1}^{n} d(y_r, y_{r+1}) \leqslant \sum_{r=1}^{n} [(d(x_{r+1}, y_r) + d(x_{r+1}, y_{r+1}))]$$

$$< 2\phi(x_1) + \Psi(y_0) + \Psi(y_1) + 2.$$

Therefore, $\{y_n\}$ is a Cauchy sequence and converges to $z_2 \in X$, so that, by Lemma 2, z_2 *is a* fixed point of F_2. By (1.3),

$\sum_{n=1}^{\infty} d(x_n, y_n)$ is convergent and $\to 0$ as $n \to \infty$

Now $d(z_1, z_2) \leqslant d(z_1, x_n) + d(x_n, y_n) + d(y_n, z_2)$.

Taking limit of both sides as $n \to \infty$, we have

$$d(z_1, z_2) = 0 \text{ and } z_1 = z_2.$$

Let F_1 be one-one and w be any fixed point of F_1. Suppose $z_1 = z_2 = z$. Then

$$H(F_1(w), F_2(z)) \leqslant \phi(w) - \phi(F_1(w)) + \Psi(z) - \Psi(F_2(z)).$$

Since $w \in F_1(w)$ and $z \in F_2(z)$, by (1.2)

$\phi(w) \leqslant \phi(F_1(w))$ and $\Psi(z) \leqslant \Psi(F_2(z))$. Hence

$H(F_1(w), F_2(z)) = 0$, so that $F_1(w) = F_2(z)$. Since this is true for any fixed point w of F_1, and z is a fixed point of F_1, we have

$$F_1(z) = F_2(z).$$

Thus $F_1(w) = F_1(z)$ and since F_1 is one-one, $w = z$. This completes the proof.

If both F_1 and F_2 are one-one, this common fixed point would be a unique common fixed point of both F_1 and F_2.

Corollary. *Let (X, d) be a complete metric space and $T = \{F_\lambda : \lambda \in \Lambda\}$ be a family of continuous multivalued mappings on X. Suppose that for each $\lambda \in \Lambda$ there exists ϕ_λ on $CB(X) \to [0, \infty)$ satisfying*

(1.5) $\phi_\lambda(A) \leqslant \phi_\lambda(B)$ $\forall$ *A, B in $CB(X)$ with $A \subset B$.*

(1.6) *Let $\exists$ a mapping $F_\alpha \in T$ which is one-one and for any other $F_\beta \in T$ $(\beta \neq \alpha)$ we have*

$$H(F_\alpha(x), F_\beta(y)) \leqslant \phi_\alpha(x) - \phi_\alpha(F_\alpha(x)) + \phi_\beta(y) - \phi_\beta(F_\beta(y))$$
for all x, y in X.

Then the mapping of T has a common fixed point which is the unique fixed point of F.

Remark 2. Theorem 1 is a generalization of the above proposition due to Felix Browder. In fact, by putting $F_1 = I_x$, $F_2 = g$, $x=y$ in Theorem 1, we obtain the above proposition.

2. A FURTHER RESULT

Now we present a result giving the condition for the existence of such a fixed point, considering a multivalued mapping.

Theorem 2. *Let (X, d) be a metric space, F be a multivalued mapping on X and ϕ be a mapping on $CB(X) \to [0, \infty)$ such that*
(2.1) $\phi(A) \leqslant \phi(B) \ \forall \ A, B$ *in* $CB(X)$ *with* $A \subset B$.

Suppose $z \in X$ be such that

(2.2) $H(F(x), z) \leqslant \phi(x) - \phi(F(x)) \ \forall \ x$ in X

then z is a fixed point of F. If F is one-one, then z is the unique fixed point of F.

Proof. Let $x_0 \in X$, then by Lamma 2, there exists an $x_1 \in F(x_0)$ such that

$$d(x_1, z) < H(F(x_0), z) + \tfrac{1}{2}$$

$$\leqslant \phi(x_0) - \phi(F(x_0)) + \tfrac{1}{2}, \text{ by } (2.2).$$

$$\leqslant \phi(x_0) - \phi(x_1) + \tfrac{1}{2}, \text{ by } (2.1)$$

Again, there exists an $x_2 \in F(x_1)$ such that

$$d(x_2, z) \leqslant H(F(x_1), z) + \frac{1}{2}$$

$$\leqslant \phi(x_1) - \phi(x_2) + \tfrac{1}{2}$$

proceeding in this manner, we construct a sequence $\{x_n\}$ such that $x_n \in F(x_{n-1})$, and

$$d(x_n, z) \leqslant \phi(x_{n-1}) - \phi(x_n) + \frac{1}{2^n}, \ n - 1, 2, 3 \ldots$$

Then

$$\sum_{r=1}^{n} d(x_r, z) \leqslant \sum_{r=1}^{n} \{\phi(x_{r-1}) - \phi(x_r) + \frac{1}{2^r}\} \leqslant \phi(x_0) + 1$$

Hence the series $\sum_{n=1}^{\infty} d(x_r, z)$ is convergent so that

$d(x_n, z) \to 0$ *as* $n \to \infty$. Hence $x_n \to z$ *as* $n \to \infty$ Since

F is continuous and $x_n \in F(x_{n-1})$, $n=1, 2, 3 \ldots$, by Lemma 2 . z is a fixed point of F .

Let F be one-one and z be any fixed point of F, Then

$H(F(z_1), z)) \leqslant \phi(z_1) - \phi(F(z)) \leqslant 0$, so that

$F(z_1) = \{z\}$ This being these of any fixed point of F, we have $F(z) = \{z\} = F(z_1)$, F being one- one, $z = z_1$. This completes the proof of Theorem 2 .

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