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APPLICATION OF GENERALISED BESSEL'S PARTIAL DIFFERENTIAL OPERATORS TO PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In Section 1 we have derived some operational formulas with the help of the partial differential operators of generalised Bessel polynomials which may be regarded as the generators of Lie algebra. In section 3 some applications of operational formulas are also pointed out .

1. INTRODUCTION

Isaac I. H. Chen and T. W. Barrett [1] have used Bessel's ordinary differential operators (which raise and lower the index of Bessel's function of the first kind) to solve some second-order linear ordinary differential equations , It may be pointed out that the operators used by Chen and Barrett are not proper Lie elements in order to generate a Lie algebra . The object of the present paper is to use the partial differential operators of M. - P. Chen and C. -C. Feng [2] in connection with the generalised Bessel polynomials, which are regarded as generators Lie algebra, in the derivation of some operational results and finally in the solution of those partial differential equations which can be factorized by means of the generators of the Lie algebra for the generalised Bessel polynomials .

In a recent paper [3] we have explained the reason why it is impossible to obtain generators of Lie algebra for generalised Bessel polynomials with two operators which raise and lower the index of generalised Bessel polynomials by unity.

In fact, from [2] we have

$$(1.1)\quad \begin{cases} \left[xy\dfrac{\partial}{\partial x} + y^2\dfrac{\partial}{\partial y} + (n-1)y \right] x_n^{\alpha} = (n+\alpha-1)\, x_n^{\alpha+1} \\[2ex] \left[x^2y^{-1}\dfrac{\partial}{\partial x} - (nx-\beta)\, y^{-1} \right] x_n^{\alpha} = \beta\, x_n^{\alpha-1}, \end{cases}$$

where $x_n^{\alpha} = y_n^{\alpha}(x)\, y^{\alpha}$, $y_n^{\alpha}(x)$ is the generalised Bessel polynomial (*cf.*, *e.g.*, [4, *p.* 75]).

If we put

$$(1.2)\quad \begin{cases} R = xy\dfrac{\partial}{\partial x} + y^2\dfrac{\partial}{\partial y} + (n-1)\, y \\[2ex] L = x^2y^{-1}\dfrac{\partial}{\partial x} - (nx-\beta)\, y^{-1} \end{cases}$$

then

$$(1.3)\quad \begin{pmatrix} 0 & R \\ L & 0 \end{pmatrix} \begin{pmatrix} x_n^{\alpha+1} \\ x_n^{\alpha} \end{pmatrix} = \begin{pmatrix} (n+\alpha-1) & x_n^{\alpha+1} \\ \beta & x_n^{\alpha} \end{pmatrix}$$

where

$$x_n^{\alpha} = y_n^{\alpha}(x)\, y^{\alpha}.$$

Also we notice that

$$(1.4)\quad LR\,[\,y_n^{\alpha}(x)\,y^{\alpha}\,] = \beta\,(\,n+\alpha-1\,)\,y_n^{\alpha}(x)\,y^{\alpha},$$

which yields the well-known relation

$$(1.5)\quad [x^2\frac{\partial^2}{\partial x^2} + xy\frac{\partial^2}{\partial x \partial y} + \beta\frac{\partial}{\partial x} - ny\frac{\partial}{\partial y} - n\,(\,n-1\,)\,]\;y_n^{\alpha}(x)\,y^{\alpha} = 0$$

Again

$$[\,R\,,L\,] = -\,\beta$$

where

$$[\,R\,,L\,] = RL - LR$$

Also we have

$$(1.6)\qquad yL = x^2\frac{\partial}{\partial x} - (\,nx - \beta\,)$$

$$xy^{-1}R - yL = xy\frac{\partial}{\partial y} - (\,\beta - x\,)$$

2. Derivation of Operational Formulas

Consider the partial differential equation

$$(2.1)\quad xy\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + (\,n-1\,)\,yu = f(\,x,y\,)$$

which is equivalent to

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + (\,n-1\,)\,u = y^{-1}f(\,x,y\,).$$

Hence the corresponding system of ordinary differential equation is

$$(2.2)\qquad \frac{dx}{x} = \frac{dy}{y} = \frac{du}{y^{-1}f(\,x,y\,) - (\,n-1\,)\,u}$$

Solving (2.2) we get $xy^{-1} = c_1$ and

(2.3) $$ux^{n-1} = c_1 \int x^{n-3} f(x, c_1^{-1} x)\, dx + c_2$$

$$= \Psi_1 (x, c_1) + c_2, \qquad (say).$$

Hence

(2.4) $$\frac{1}{R}\Big[f(x, y) \Big] = x^{1-n} \Psi_1 (x, c_1) \Big|_{c_1 = xy^{-1}} + x^{1-n} \phi_1 (xy^{-1}).$$

where Ψ_1 is given in (2.3) anc ϕ_1 is arbitrary.

Corollary 1. If $y^{-1} f(x, y)$ = a function of x only $= P(x)$, (*say*), then

(2 5) $$\frac{1}{R}\Big[f(x, y) \Big] = x^{1-n} \int x^{n-2} P(x)\, dx + x^{1-n} \phi_1 (xy^{-1}),$$

Where ϕ_1 is arbitrary.

Corollary 2. If $y^{-1} f(x,y)$ = a function of y only $= Q(y)$ (*say*), then

(2.6) $$\frac{1}{R}\Big[f(x, y) \Big] = y^{1-n} \int y^{n-2} Q(y)\, dy + y^{1-n} \phi_1 (xy^{-1}),$$

where ϕ_1 is arbitrary.

Next, consider the partial differential equation

(2.7) $$x^2 y^{-1} \frac{\partial u}{\partial x} - (nx-\beta) y^{-1} u = F(x, y),$$

which is equivalent to

$$x^2 \frac{\partial u}{\partial x} - (nx - \beta) u = y F(x, y).$$

Hence the corresponding system of ordinary differential equation is

$$(2.8)\quad \frac{dx}{x^2} = \frac{dy}{0} = \frac{du}{yF(x, y) + (nx-\beta)u}$$

Solving (2.8) , we get $y = c_1$ and

$$(2.9)\quad u\,(e^{-\beta/x}\,x^{-n}) = c_1 \int \frac{F(x, c_1)\,dx}{x^{n+2}\,e^{\beta/x}} + c_2$$

$$= \Psi_2(x, c_1) + c_2 \quad (say)$$

Hence

$$(2.10)\quad \frac{1}{L} F(x, y) = e^{\beta/x}\,x^n\,\Psi_2(x, c_1)\Big|_{c_1=y} + e^{\beta/x}\,x^n\,\phi_2(y)$$

where Ψ_2 is given in (2.9) and ϕ_2 is arbitrary .

COROLLARY 8. *If* $yF(x,y)$ = a function of x only=P (x), (*say*), then

$$(2.11)\quad \frac{1}{L} F(x, y) = e^{\beta/x}\,x^n \int \frac{P(x)}{x^{n+2}\,e^{\beta/x}}\,dx + e^{\beta/x}\,x^n\,\phi_2(y)$$

where ϕ_1 is arbitrary .

COROLLARY 4. *If* $yF(x,y)$=a function of y only = $Q(y)$, (*say*), then there exists no solution of u .

3. Application of the Operational Formulas

Consider the partial differential equation

$$(3.1)\quad x^3\frac{\partial^2 u}{\partial x^2} + x^2 y\,\frac{\partial^2 u}{\partial x \partial y} + \beta x\,\frac{\partial u}{\partial x} - (nx-\beta)\,y\frac{\partial u}{\partial y} - \{n\,(nx-\beta) - (nx-2\beta)\}\,u = f(x,y) .$$

Since

$$[\, xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (n-1)\, y \,]\, [\, x^2 y^{-1} \frac{\partial}{\partial x} - (nx-\beta)\, y^{-1} \,]$$

$$= x^3 \frac{\partial^2}{\partial x^2} + x^2 y \frac{\partial^2}{\partial x \partial y} + \beta x \frac{\partial}{\partial x} - (nx-\beta)\, y \frac{\partial}{\partial y} - \{\, n\{nx-\beta\} - (nx-2\beta) \,\}.$$

The above equation (3.1) becomes

$$RLu = f(x, y) .$$

It follows from (2.4) that

$$(3.2) \quad Lu = x^{1-n}\, \Psi_1 (x, c_1) \Big|_{c_1 = xy^{-1}} + x^{1-n}\, \phi_1 (\, xy^{-1} \,)$$

$$(3.3) \quad = F\,(x,y)\,,\ (say),$$

where Ψ_1 is given in (2.3) and ϕ_1 is arbitrary . Again it follows from (2.10) that

$$(3.4) \quad u = e^{\beta/x}\, x^n\, \Psi_2 (\, x, c_1 \,) \Big|_{c_1 = y} + e^{\beta/x}\, x^n\, \phi_2 (y)$$

where Ψ_2 is given in (2.9) and ϕ_2 is arbitrary .

On the other hand, if we consider the equation

$$LRu = F(\, x, y \,),$$

which is equivalent to

$$(3.5) \quad x^3 \frac{\partial^2 u}{\partial x^2} + x^2 y \frac{\partial^2 u}{\partial x \partial y} + \beta x \frac{\partial u}{\partial x} - (\, nx-\beta \,)\, y \frac{\partial u}{\partial y}$$

$$- (\, nx-\beta \,)\, (\, n-1 \,)\, u = F(\, x, y \,),$$

then it follows from (2.10) that

$$(3.6) \qquad Ru = e^{\beta/x} x^n \Psi_2(x, c_1)\Big|_{c_1=y} + e^{\beta/x} x^n \phi_2(y)$$

$$(3.7) \qquad = f(x, y), \text{ say},$$

where Ψ_2 is given in (2.9) and ϕ_2 is arbitrary. Hence, from (2.4) it follows that

$$(3.8) \qquad u = x^{1-n} \Psi_1(x, c_1)\Big|_{c_1 = xy^{-1}} + x^{1-n} \phi_1(xy^{-1})$$

where Ψ_1 is given in (2.3) and ϕ_1 is arbitrary.

REFERENCES

[1] Isaac I. H. Chen and T. W. Barrett, Besrel's differential operators and application to linear differential equations, *Internat, J Math, Ed. Sci. Tech.* 13 (1982), 140-153 .

[2] M. - P. Chen and C. - C. Feng, Group theoretic origins of certain generating functions of generalized Bessel polynomials, Tamkang *J. Math.* 6 (1975) , 87 - 93 .

[3] B. Ghosh, Derivation of raising and lowering operators of generalised Bessel polynomials, *Math. Ed.* (*Siwan*) 20 (1986) 20 - 22 .

[4] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1984.