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ON A CHAIN INVOLVING THE H-FUNCTION TRANSFORM

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ABSTACRT

In this paper we establish a new chain interconnecting a number of H-function transforms having different parameters. Full care has been taken to give an account of all the convergence and existance conditions for the validity of the chain. Since the H-function is very general in nature, the corresponding chains for a number of other integral transforms can easily be obtained as special cases of our chain. Similar chains for various simpler integral transforms have been obtained earlier by P, N. Rathie [7], U. C. Jain [6], S. P. Goyal [2], K. C, Gupta and P. K. Mittal [5],

1. INTRODUCTION

Throughout this work we use the notation

$$\phi(s) = H\left[f(t) : s : {}^{m,\, n}_{p,\, q} : {}^{(a_j,\, \alpha_j)_{1,p}}_{(b_j,\, \beta_j)_{1,q}} \right]$$

$$= \int_0^\infty H_{p,\, q}^{m,\, n} \left[st \,\middle|\, \begin{matrix} (a_j,\, \alpha_j)_{1,p} \\ (b_j,\, \beta_j)_{1,q} \end{matrix} \right] f(t)\, dt \qquad \ldots (1.1)$$

for the H-function transform defined by Gupta and Mittal [4.*p.* 142]. The Kernel in (1.1) is the well-known H-function which is quite general in nature and includes most of the commonly used funetions as its special cases. The definition of this H-function in terms of a Mellin Barnes contour integral, various conditions satisfied by its parameters for the convergence of the defining integral, some of its special cases, important properties, and asymptoic expansions, can be referred to in a standard work on the subject by Srivastava, Gupta, and Goyal [8, pp. 10-14]. It has been assumed that the various H-functions occurring in this paper satisfy the conditions corresponding appropriately to those given by (2 2.11(on page 13 in the book referred to above. On account of the general nature of the kernel, this transform provides us with interesting unification and generalizations of a numer of integral transforms (see [8, Chapter 4] for details) .

2. MAIN THEOREM

THEOREM. *If*

$$\phi_1(s) = H\left[f(t) : s : \begin{matrix} m_1,\, n_1 \\ p_1,\, q_1 \end{matrix} : \begin{matrix} (a_j^{(1)}, \alpha_j^{(1)})_{1,p_1} \\ (b_j^{(1)}, \beta_j^{(1)})_{1,q_1} \end{matrix} \right], \qquad \ldots (2.1)$$

and

$$\phi_k(s) = H\left[t^{c_k - 1}\, \phi_{k-1}\left(\frac{1}{t}\right) : s : \begin{matrix} m_k,\, n_k \\ p_k,\, q_k \end{matrix} : \begin{matrix} (a_j^{(k)}, \alpha_j^{(k)})_{1,p_k} \\ (b_j^{(k)}, \beta_j^{(k)})_{1,q_k} \end{matrix} \right] \qquad \ldots (2.2)$$

$k = 2,3, \ldots, N)$,

then

$$\phi_N(s) = s^{-\sum_{i=1}^{N-1} c_i - (N-1)} \int_0^\infty H^{\sum_{i=1}^{N} m_1, \sum_{i=1}^{N} n_i}_{\sum_{i=1}^{N} p_1, \sum_{i=1}^{N} q_i} \left[st \,\middle|\, \begin{matrix} I_N \\ J_N \end{matrix} \right] f(t)\, dt \quad \ldots (2.3)$$

where N is a positive integer greater than 1,

$$I_N = [(a_j^{(M)} + \{ \sum_{i=1}^{M-1} c_i + (M-1) \} \alpha_j^{(M)}, \alpha_j^{(M)})_{1,n_M}]_{M=1}^{N-1},$$

$$(a_j^{(N)} + \{ \sum_{i=1}^{N-1} c_1 + (N-1) \} \alpha_j^{(N)}, \alpha_j^{(N)})_{1.p_N},$$

$$[(a_j^{(M)} + \{ \sum_{i=1}^{M-1} c_i + (M-1) \} \alpha_j^{(M)}, \alpha_j^{(M)})_{n_M+1,p_M}]_{M=1}^{N-1}$$

and

$$J_N = [(b_j^{(M)} + \{ \sum_{i=1}^{M-1} c_i + (M-1) \} \beta_j^{(M)}, \beta_j^{(M)})_{1, m_M}]_{M=1}^{N-1}.$$

$$(b_j^{(N)} + \{ \sum_{i=1}^{N-1} c_i + (N-1) \} \beta_j^{(N)}, \beta_j^{(N)})_{1,q_N},$$

$$[(b_j^{(M)} + \{ \sum_{i=1}^{M-1} c_i + (M-1) \} \beta_j^{(M)}, \beta_j^{(M)})_{m_M + 1, q_M}]_{M=1}^{N-1},$$

it being assumed that the integrals involved in equations (2.1) *and* (2.2) *are absolutely convergent and the following sets of conditions are satisfied :*

$A_N > 0,\ A'_N > 0,\ |\arg(s)| < \frac{1}{2} A'_N \pi$,

$$Re\left[\sum_{i=1}^{N-1} c_i + (N-1)\right] - \sum_{M=1}^{N-1} \max_{1\leqslant j\leqslant n_M} \left[Re \frac{a_j^{(M)} + \{\sum_{i=1}^{M-1} c_i + (M-1)\}\alpha_j^{(M)} - 1}{\alpha_j^{(M)}} \right] +$$

$$+ \min_{1\leqslant j\leqslant m_N} Re \frac{b_j^{(N)}}{\beta_j^{(N)}} > 0,$$

$$Re\left[\sum_{i=1}^{N-1} c_i + (N-1)\right] - \sum_{M=1}^{N-1} \min_{1\leqslant j\leqslant m_N} \left[Re \frac{b_j^{(M)} + \{\sum_{i=1}^{M-1} c_i + (M-1)\}\beta_j^{(M)}}{\beta_j^{(M)}} \right] +$$

$$+ \max_{1\leqslant j\leqslant n_N} Re \frac{a_j^{(N)} - 1}{\alpha_j^{(N)}} < 0,$$

where

$$A_N = \sum_{M=1}^{N-1} \left[\sum_{j=1}^{n_M} \alpha_j^{(M)} - \sum_{j=n_M+1}^{p_M} \alpha_j^{(M)} + \sum_{j=1}^{m_{(M)}} \beta_j^{(M)} - \sum_{j=m_M+1}^{q_M} \beta_j^{(M)} \right],$$

$$A'_N = \sum_{j=1}^{n_N} \alpha_j^{(N)} - \sum_{j=n_N+1}^{p_N} \alpha_j^{(N)} + \sum_{j=1}^{m_N} \beta_j^{(N)} - \sum_{j=m_N+1}^{q_N} \beta_j^{(N)}$$

and the notation $[(a_j^{(M)}, \alpha_j^{(M)})_{1,\,u_M}]_{M=1}^{N}$ *stands for a sequence of*

$$\sum_{i=1}^{N} u_i \text{ pairs } (a_j^{(1)}, \alpha_j^{(1)})_{1,u_1}, (a_j^{(2)}, \alpha_j^{(2)})_{1,u_2}, \ldots, (a_j^{(N)}, \alpha_j^{(N)})_{1,u_n}.$$

Proof . Interpreting (2.2) for $k=2$ with the help of (1.1), we get

$$\phi_2(s) = \int_0^\infty x^{c_1} H_{p_2,q_2}^{m_2,n_2}\left[sx \left| \begin{matrix} (a_j^{(2)}, \alpha_j^{(2)})_{1,p_2} \\ (b_j^{(2)}, \beta_j^{(2)})_{1,q_2} \end{matrix} \right.\right] \phi_1\left(\frac{1}{x}\right) dx \qquad \ldots(2\ 4)$$

Substituting the value of $\phi_1(1/x)$ from (2.1) in (2.4), we get

$$\phi_2(s) = \int_0^\infty x^{c_1} H_{p_2,q_2}^{m_2,n_2}\left[sx \left| \begin{matrix} (a_j^{(2)}, \alpha_j^{(2)})_{1,p_2} \\ (b_j^{(2)}, \beta_j^{(2)})_{1,q_2} \end{matrix} \right.\right]$$

$$\cdot \left\{ \int_0^\infty H_{p_1,q_1}^{m_1,n_1}\left[tx^{-1} \left| \begin{matrix} (a_j^{(1)}, \alpha_j^{(1)})_{1,p_1} \\ (b_j^{(1)}, \beta_j^{(1)})_{1,q_1} \end{matrix} \right.\right] f(t)\, dt \right\} dx \qquad \ldots(2.5)$$

On inverting the order of integrations in (2.5) which is easily seen to be permissible under the conditions stated with the theorem we get

$$\phi_2(s) = \int_0^\infty \left\{ \int_0^\infty x^{c_1} H_{p_1,q_1}^{m_1,n_1}\left[tx^{-1} \left| \begin{matrix} (a_j^{(1)}, \alpha_j^{(1)})_{1,p_1} \\ (b_j^{(1)}, \beta_j^{(1)})_{1q_1,} \end{matrix} \right.\right]\right.$$

$$\left. \cdot\, H_{p_2,q_2}^{m_2,n_2}\left[sx \left| \begin{matrix} (a_j^{(2)}, \alpha_j^{(2)})_{1,p_2} \\ (b_j^{(2)}, \beta_j^{(2)})_{1,q_2} \end{matrix} \right.\right] dx \right\} f(t)\, dt \qquad \ldots(2.6)$$

Evaluating the x-integral involved in (2.6) with the help of modified form of a result [3, p. 601], we get

$$\phi_2(s) = \int_0^\infty s^{-c_1-1} H^{m_1+m_2,\, n_1+n_2}_{p_1+p_2,\, q_1+q_2}\left[ts \,\middle|\, \begin{matrix} I_2 \\ J_2 \end{matrix} \right] f(t)\, dt \qquad \ldots (2.7)$$

where

$$I_2 = [(a_j^{(1)}, \alpha_j^{(1)})_{1,n_1}, (a_j^{(2)} + (c_1+1)\, \alpha_j^{(2)}, \alpha_j^{(2)})_{1,p_2},$$

$$(a_j^{(1)}, \alpha_j^{(1)})_{n_1+1,\, p_1}]$$

and

$$J_2 = [(b_j^{(1)}, \beta_j^{(1)})_{1,m_1}, (b_j^{(2)} + (c_1+1)\, \beta_j^{(2)}\, \beta_j^{(2)})_{1,q_2},$$

$$(b_j^{(1)}, \beta_j^{(1)})_{m_1+1,\, q_1}]$$

and the following set of conditions are assumed to be satisfied :

$A_2 > 0$, $A'_2 \geq 0$, $|\arg(s)| < \frac{1}{2} A'_2 \pi$

$$Re\,(c_1+1) - \max_{1 \leqslant j \leqslant m_1} \left[Re \frac{a_j^{(1)}-1}{a_j^{(1)}} \right] + \min_{1 \leqslant j \leqslant m_2} \left[Re \frac{b_j^{(2)}}{\beta_j^{(2)}} \right] > 0,$$

$$Re\,(c_1+1) - \min_{1 \leqslant j \leqslant m_1} \left[Re \frac{b_j^{(1)}}{\beta_j^{(1)}} \right] + \max_{1 \leqslant j \leqslant n_2} \left[Re \frac{a_j^{(2)}-1}{\alpha_j^{(2)}} \right] < 0,$$

where

$$A_2 = \sum_{j=1}^{n_1} \alpha_j^{(1)} - \sum_{j=n_1+1}^{p_1} \alpha_j^{(1)} + \sum_{j=1}^{m_1} \beta_j^{(1)} - \sum_{j=m_1+1}^{q_1} \beta_j^{(1)}$$

and

$$A'_2 = \sum_{j=1}^{n_2} \alpha_j^{(2)} - \sum_{j=n_2+1}^{p_2} \alpha_j^{(2)} + \sum_{j=1}^{m_2} \beta_j^{(2)} - \sum_{j=m_2+1}^{q_2} \beta_j^{(2)}.$$

Now, by assuming the result (2.3) to be true for $N=K$ and proceeding in a manner similar to the above, we can easily arrive at the corresponding result for $N=K+1$. Much of the computational work involved in proving this is straightforward, so we omit the details. Also, from (2.1) and (2.7), we find that the result (2.3) is true for $N=2$. Thus by mathematical induction, we can say that the result (2.3) is true for every positive integer N greater than 1.

3. SPECIAL CASES

1· If in the main theorem, we replace m_k by m_k+1, p_k by m_k, q_k by m_k+1, n_k by 0, $a_j^{(k)}$ by $a_j^{(k)}+b_j^{(k)}$ ($j=1,2,\ldots, m_k, k=1,2,\ldots,N$) and take $\alpha_j^{(k)}=\beta_j^{(k)}=1$ ($j=1,2,\ldots, m_k+1$; $k=1,2,\ldots,N$), $b^{(k)}_{m_k+1}=\rho_k$ therein, it reduces to the following result involving the G-function transforms (see [1] and [8, chapter 4]) :

COROLLARY 1. *If*

$$\phi_1(s)=G\left[f(t):s:\begin{matrix}m_1+1,0\\ m_1,m_1+1\end{matrix}:\begin{matrix}(a_j^{(1)}+b_j^{(1)})_{1,m_1}\\ (b_j^{(1)})_{1,m_1},\rho_1\end{matrix}\right] \quad \ldots (3.1)$$

and

$$\phi_k(s)=G\left[t^{c_{k-1}}\phi_{k-1}\left(\frac{1}{t}\right):s:\begin{matrix}m_k+1,0\\ m_k,m_k+1\end{matrix}:\begin{matrix}(a_j^{(k)},+b_j^{(k)})_{1,m_k}\\ (b_j^{(k)})_{1,m_k},\rho_k\end{matrix}\right] \ldots (3.2)$$

$$(k=2,3,\ldots,N),$$

then

$$g_N(s) = s^{-\sum_{i=1}^{N-1} c_i-(N-1)} \int_0^\infty G^{\sum_{i=1}^{N} m_i+N,0 \quad n_i}_{\sum_{i=1}^{N} m_i, \sum_{i=1}^{N} m_i+N}\left[st \,\middle|\, \begin{matrix} P_N \\ Q_N \end{matrix}\right] f(t)\, dt$$

... (3.3)

where

$$P_N = [(a_j^{(M)} + b_j^{(M)} + \sum_{i=1}^{M-1} c_i + M - 1)_{1,\, m_M}]_{M=1}^{N},$$

$$Q_N = [(b_j^{(M)} + \sum_{i=1}^{M-1} c_i + M-1)_{1,\, m_M}]_{M=1}^{N},$$

$$[(\rho_M + \sum_{i=1}^{M-1} c_i + M - 1)]_{M=1}^{N}$$

and the conditions of validity directly obtainable from the main theorem are assumed to be satisfied .

II. If, in Corollary 1, we take $m_k = \rho_k = 0$ ($k = 1, 2, \ldots, N$), it reduces to the following result involving the well-known Laplace transforms :

COROLLARY 2. *If*

$$\phi_1(s) = L [f(t) ; s]$$

and , (3.4)

$$\phi_k(s) = L [t^{C_k-1} \phi_{k-1} (\tfrac{1}{t}) ; s] , (k= 2, 3, \ldots, N),$$

...(3.5)

then

$$\phi_N(s) = s^{-\sum_{i=1}^{N-1} c_i - (N-1)} \int_0^\infty H_{0,N}^{N,0}\left[st \left| \begin{matrix} - \\ R_N \end{matrix} \right. \right] f(t)\, dt \qquad .., (3.6)$$

where $R_N = \left(\sum_{i=1}^{M-1} c_i + M-1 \right)_{M=1}^{N}$ *and the conditions of validity directly obtainable from the main theorem are assumed to be satisfied,*

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