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ON EXPANSIONS OF q-SERIES OF THREE VARIABLES

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1. INTRODUCTION

Pathan [1] establishd certain expansions of hypergeometric functions of three variables. Recently, Singhal and his co-worker [2] pointed out that all the above expansions are immediate consequence of the simple binomial identity

$$(1.1)\qquad \sum_{r=0}^{n} \binom{n}{r} (a)_{r+k} (c-a)_{n-r} = (a)_k \ (c+k)_n$$

and used it to establish expansions of certain hypergeometric functions of three variables due to Srivastava [4].

Here we make use of the following basic identity to show how it could be utilized to establish expansion of q-series of three variables, thus, extending the above known expansions to their basic analogues :

$$(1.2)\qquad \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} [a]_{r+k} [c/a]_{n-r} \, a^{n-r} \, q^{k(n-r)} = [a]_k \, [cq^k]_n ,$$

where $\qquad \begin{bmatrix} n \\ r \end{bmatrix} = \dfrac{[q]_n}{[q]_r [q]_{n-r}}$

and $[\alpha]_n = (1-\alpha)(1-\alpha q) \ldots, (1-\alpha q^{n-1}), [\alpha]_0 = 1$.

The proof of (1.2) is simple, ond we omit the details .

Following Srivastava [5], we define a general q-hypergeometric series of three variables by

$$(1.3)\quad \phi^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');x,y,z\\(e)::(f);(f');(f''):(g);(g');(g'');\lambda,\mu,\nu\end{matrix}\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{[(a)]_{m+n+p}[(b)]_{m+n}[(b')]_{n+p}[(b'')]_{p+m}[(c)]_m[(c')]_n}{[(e)]_{m+n+p}[(f)]_{m+n}[(f')]_{n+p}[(f'')]_{p+m}[(g)]_m[(g')]_n}$$

$$\cdot \frac{[(c'')]_p\, x^m y^n z^p q^{\{\lambda m(m+1)+\mu n(n+1)+\nu p(p+1)\}/2}}{[(g'')]_p\, [q]_m\, [q]_n\, [q]_p}$$

where λ, μ and ν are positive to ensure rapid convergence of the series, In case $\lambda=0=\mu=\nu$, $|x|<1$, $|y|<1$ and $|z|<1$.

Also, (a) shall stand for the sequence of A parameters $a_1, a_2, \ldots, a_A$. In what follows, the other symbols will have their usual meaning.

2, EXPANSION FORMULAS

Here we show how (1.2) can be utilized to establish the q-series expansions . To show this, we establish the following expansion

$$(2.1)\quad \sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{k=0}^{p} \begin{bmatrix}m\\r\end{bmatrix}\begin{bmatrix}n\\s\end{bmatrix}\begin{bmatrix}p\\k\end{bmatrix} \frac{[u]_r\,[v/u]_{m-r}\,[u']_{n-s}\,[q/v']_s}{[v]_m\,[qu/v']_n\,[q/u'']_p}$$

$$\cdot\, [q/v'']_k\,[v''/u'']_{p-k}\, u^{m-r}(q/v')^{n-s}(q/v'')^{p-k}$$

$$\cdot\ \phi^{(3)}\left[\begin{matrix}(a)::(b);\ (b');(b''):uq^{r},\ (c);\ u'q^{n-s},\ (c');\ u''q^{-p},\ (c'');\ xq^{m-r},\\ ,\ yq^{n},z\\ (e)::(f);(f');(f''):vq^{m},\ (g);\ v'q^{s},\ (g');\ v''q^{-k};\ \lambda,\mu,\nu\end{matrix}\right]$$

$$=\phi^{(3)}\left[\begin{matrix}(a)::(b);\ (b');\ (b''):u,\ (c);\ u',\ (c');\ u'',\ (c'');\ x,y,z\\ (e)::(f);\ (f');\ (f''):v,\ (g');\ v'',\ (g'');\ \lambda,\mu,\nu\end{matrix}\right].$$

To prove (2.1), we replace the $\phi^{(3)}$-series by its triple series given by (1.3), interchange the order of summations (valid under the given conditions), and evaluate the inner sums with the help of (1.2), and we get the right side of (2.1) after some simplification .

3. SPECIAL CASES

If we let $q\to 1$ in (2 1) after making some parametaic changes, we get a known expansion due to Singhal and Soni [1,*p*. 38 (1.3)] . Thus, we can also establish the basic analogues of the other expansions due to Singhal and Soni [2]. These would, in turn, include the basic analogues of the expansions given earlier by Pathan [1] .

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