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SOME MULTIPLIERS FOR THE | C, 1 | SUMMABILITY OF LAPLACE SERIES

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ABSTRACT

In the present paper a new theorem on the absolute $(C,1)$ summability of Laplace series has been proved .

1. INTRODUCTION

Let $f(\theta,\phi)$ be a function defined for the range $0\leqslant\theta\leqslant\pi, 0\leqslant\phi\leqslant 2\pi$ and integrable (L) over the surface of the sphere S. The Laplace series associated with this function is

$$f(\theta,\phi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) \iint_S f(\theta',\phi')\, P_n(\cos w)\, dS', \tag{1.1}$$

where

$$\cos w = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$
$$dS' = \sin\theta'\, d\theta'\, d\phi' .$$

The Legendre polynomials $P_n(x)$ are defined by the relation

$$(1-2\mu x+\mu^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\,\mu^n . \tag{1.2}$$

Following Kogbetliantz [3] the general value of $f(\theta,\phi)$ is

$$\phi(w) = \frac{1}{2\pi\sin w} \int_{C\omega} f(\theta',\phi')\, ds' , \tag{1.3}$$

where the curvilinear integral is taken along the small circle C_w, whose centre is (θ, ϕ) on the sphere and whose curvilinear radius is w. In view of the relation (1.3) the series (1.1) reduces to the form

$$\sum_{n=0}^{\infty} (n+\tfrac{1}{2}) \int^{\pi} \phi(w)\, P_n(\cos w) \sin w \, dw \equiv \sum_{n=0}^{\infty} A_n . \qquad (1.4)$$

We write

$$\phi(t) = \int_t^{\delta} w^{-1/2} \, | \phi(w) | \, dw, \qquad 0<\delta\leqslant\pi$$

$$\beta_n = \prod_{\mu=1}^{k-1} (\log^{\mu} n)(\log^{k} n)^{1+\epsilon}, \quad \epsilon>0$$

where

$$\log^k n = \log(\log^{k-1} n), \quad .\,, \log^2 n = \log \log n,$$

and prove the following

THEOREM. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\sum_{n=1}^{\infty} \lambda_n / n < \infty$, *and if*

$$\phi(t) = O\{ t^{\frac{1}{2}} (\log^k (1/t))^{\eta} \} \text{ as } t \to +0, \qquad (1.5)$$

then the series

$$\Sigma \lambda_n A_n / \beta_n$$

is summable $| C, 1 |$ *for* $0<\eta<\epsilon$.

2. RESULTS REQUIRED

The following lemmas are required in our investigation :

Lemma 1.

$$| P_n(\cos\theta) | = O(1) \text{ for } 0\leqslant\theta\leqslant\pi .$$

Lemma 2 [1] .

$$-\frac{\pi}{2}\left[\frac{d}{dx}\{P_{n+1}(x)+P_n(x)\}_{x=\cos w}\right]\sin w$$

$$=R\Big\{(n+1)e^{i(n+1)w+\pi/4}[\cot\tfrac{1}{2}w]^{\frac{1}{2}}\int_0^1(1-u)^n u^{-\frac{1}{2}}g(w,u)du$$

$$-(n+1)e^{i(n+3/2)w+\pi/4}(2\sin w)^{-\frac{1}{2}}\int_0^1(1-u)^n u^{-\frac{1}{2}}g(w,u)du$$

$$+e^{i(n+3/2)w+\pi/4}(2\sin w)^{-\frac{1}{2}}\int_0^1(1-u)^{n+1}u^{-\frac{1}{2}}g(w,u)du$$

$$+\tfrac{1}{2}e^{i(n+2)w+3\pi/4}(\sin\tfrac{1}{2}w)^{-1}(2\sin w)^{-\frac{1}{2}}\int_0^1(1-u)^{n+1}$$

$$u^{-\frac{1}{2}}[g(w,u)]^3\,du$$

$$-e^{i(n+5/2)w+3\pi/4}(2\sin w)^{-3/2}\int_0^1(1-u)^{n+1}u^{\frac{1}{2}}[g(w,u)]^3\,du\Big\},$$

where

$$g(w,u)=\{1-ue^{iw}(2i\sin w)^{-1}\}^{-\frac{1}{2}}\leqslant\sqrt{2}.$$

Now let this be equal to $J_1+J_2+J_3+J_4+J_5$, say, where

$$J_1=A\frac{(n+1)\Gamma(n+1)}{\Gamma(n+3/2)}(\cot\tfrac{1}{2}w)^{\frac{1}{2}}e^{i(n+1)w}$$

$$J_2=A\frac{(n+1)\Gamma(n+1)}{\Gamma(n+5/2)}(\sin w)^{-\frac{1}{2}}e^{i(n+3/2)w}$$

$$J_3 = A \frac{\Gamma(n+2)}{\Gamma(n+5/2)} (\sin w)^{-\frac{1}{2}} e^{i(n+3/2)w}$$

$$J_4 = A \frac{\Gamma(n+2)}{\Gamma(n+5/2)} (\cos \tfrac{1}{2}w)^{-\frac{1}{2}} (\sin \tfrac{1}{2}w)^{-3/2} e^{i(n+5/2)w}$$

$$J_5 = A \frac{\Gamma(n+2)}{\Gamma(n+,/2)} (\sin w)^{-5/2} e^{i(n+5/2)w} ,$$

and A is a constant, not necessary the same at each occurrence.

3. PROOF OF THE THEOREM

Let τ_n be the *nth* Cesàro mean of first order of the sequence $\{ n\lambda_n A_n/\beta_n \}$. It is sufficient to prove that

$$\sum_{n=2}^{\infty} |\tau_n| / n < \infty .$$

$$\tau_n = \frac{1}{n+1} \int_0^{\pi} \phi(w) \left\{ \sum_{\nu=2}^{n} (\nu+\tfrac{1}{2}) P_\nu(\cos w) \sin w \,.\, \nu\lambda_\nu \beta_\nu^{-1} \right\} dw$$

$$= \int_0^{1/n} + \int_{1/n}^{\pi-(1/n)} + \int_{\pi-(1/n)}^{\pi}$$

$$= I_1 + I_2 + I_3, \text{ say}.$$

$$| I_1 | = 0(n^{-1}) \int_0^{1/n} w \, | \phi(w) | \, dw \,.\, \sum_{\nu=2}^{n} \nu(\nu+\tfrac{1}{2}) \lambda_\nu \beta_\nu^{-1}$$

Now

$$\int_0^t | \phi(w) | \, dw = \int_0^t w^{\frac{1}{2}} \phi'(w) \, dw$$

$$= \Big[\; w^{\frac{1}{2}} \;\; \phi\,(w) \; \Big]_0^t - \frac{1}{2} \int_0^t w^{-\frac{1}{2}} \phi\,(w)\, dw$$

$$= O \Big[w\,(\, log^k\,(\,1/w\,)\,)^\eta \;\Big]_0^t - O \Big\{ \int_0^t \Big[w^\alpha\,(\,log^k\,(\,1/w\,)\,)^\eta \;\Big] w^{-\alpha}\, dw \Big\}\,,$$

$$0 < \alpha < 1$$

$$= O\,\{\; t\,(\, iog^k\,(\,1/t\,)\,)^\eta \,\}\,.$$

Hence

$$|\,I_1\,| = O\,(\,n\,\beta_n{}^{-1}\,) \int_0^{1/n} |\,\phi(w)\,|\; dw \quad \sum_{\nu=2}^{n} \frac{\lambda_\nu}{\nu}$$

$$= O\;\{\,(\,log^k n\,)^\eta\,/\beta_n\,\}\;. \tag{3.1}$$

By Abel's transformation,

$$I_2 = \frac{1}{n+1} \int_{1/n}^{\pi-(1/n)} \phi\,(w) \Big\{ \sum_{\nu=0}^{n-1} \; \sum_{r=0}^{\nu} \;(\,r+\tfrac{1}{2}\,)\, P_r(\,cos\,w\,)\, sin\, w\grave{a}.$$

$$\triangle\,(\,\nu\lambda_\nu\beta_\nu{}^{-1}\,) + \sum_{\nu=0}^{n}\,(\,\nu+\tfrac{1}{2}\,)\; P_\nu\,(\,cos\; w\,)\; sin\; w\;.\,n\;\lambda_n\;\beta_n{}^{-1}\;\Big\}\,dw$$

$$= \frac{1}{n+1} \int_{1/n}^{\pi-(1/n)} \phi\,(w\,) \Big\{ \sum_{\nu=0}^{n-1} \;\frac{d}{dx} \Big[\, P_{\nu+1}\,(x) \;+\; P_\nu\,(x) \;\Big]_{x=cos\,w}$$

$$sin\; w\,.\; \triangle\,(\,\nu\lambda_\nu\,\beta_\nu{}^{-1}\,)$$

$$+ \;\frac{d}{dx} \Big[P_{n+1}\,(x) + P_n\,(x) \Big]_{x=cos\,w} \; sin\; w.n\lambda_n\alpha_n{}^{-1} \Big\}\, dw$$

$$= I_{2,1} + I_{2,2}\;,\; say\;.$$

$$I_{2,1} = \frac{1}{n+1} \int_{1/n}^{\pi-(1/n)} \phi(w) \left\{ \sum_{\nu=0}^{n-1} [\, J_1+J_2+J_3+J_4+J_5 \,] \Delta(\nu\lambda_\nu\beta_\nu^{-1}) \right\} dw$$

$$= I_{2,1,1} + I_{2,1,2} + I_{2,1,3} + I_{2,1,4} + I_{2,1,5}, \text{ say}.$$

As the other cases are similar, we consider only $I_{2,1,1}$ and $I_{2,1,4}$.

Since

$$\Delta(\nu\lambda_\nu\beta_\nu^{-1}) = \nu\lambda_\nu \Delta\beta_\nu^{-1} + \nu\beta_{\nu+1}^{-1}\Delta\lambda_\nu - \lambda_{\nu+1}\beta_{\nu+1}^{-1}$$

$$= O(\lambda_\nu\beta_\nu^{-1}) + O(\nu\beta_\nu^{-1}\Delta\lambda_\nu)$$

therefore

$$|I_{2,1,1}| = O(n^{-1}) \int_{1/n}^{\pi-(1/n)} w^{-\frac{1}{2}} |\phi(w)|\, dw \left\{ \sum_{\nu=0}^{n-1} \nu^{\frac{1}{2}} \left[O(\lambda_\nu\beta_\nu^{-1}) + O(\nu\beta_\nu^{-1}\Delta\lambda_\nu) \right] \right\}$$

$$= O\{n^{-3/2}(\log^k n)^\eta\} \left\{ O(n^{3/2}\beta_n^{-1}) \sum_{\nu=0}^{n-1} \frac{\lambda_\nu}{\nu} + O(n^{3/2}\beta_n^{-1}) \sum_{\nu=0}^{n-1} \Delta\lambda_\nu \right\}$$

$$= O\{(\log^k n)^\eta/\beta_n\}$$

and

$$|I_{2,1,4}| = O(n^{-1}) \int_{1/n}^{\pi-(1/n)} w^{-3/2} |\phi(w)|\, dw \left\{ \sum_{\nu=0}^{n-1} \nu^{-\frac{1}{2}} \left[O(\lambda_\nu\beta_\nu^{-1}) + O(\nu\beta_\nu^{-1}\Delta\lambda_\nu) \right] \right\}$$

$$= O\left\{ n^{-\frac{1}{2}} (\log^k n)^\eta \right\} \cdot \left\{ O(n^{\frac{1}{2}}\beta_n^{-1}) \sum_{\nu=0}^{n-1} \frac{\lambda_\nu}{\nu} + 0(n^{\frac{1}{2}}\beta_n^{-1}) \sum_{\nu=0}^{n-1} \Delta\lambda_\nu \right\}$$

$$= O\{(\log^k n)^\eta/\beta_n\}.$$

Hence

$$| I_{2,1} | = O \{ (\log^k n)^{\eta} / \beta_n \} .$$

$$I_{2,2} = \frac{n \lambda_n \beta_n^{-1}}{n+1} \int_{1/n}^{\pi-(1/n)} \phi(w) [J_1 + J_2 + J_3 + J_4 + J_5] \, dw$$

$$= I_{2,2,1} + I_{2,2,2} + I_{2,2,3} + I_{2,2,4} + I_{2,2,5} , \text{ say} .$$

$$| I_{2,2,1} | = O(n^{1/2} \lambda_n \beta_n^{-1}) \int_{1/n}^{\pi-(1/n)} w^{-\frac{1}{2}} | \phi(w) | \, dw = O\left\{ \lambda_n (\log^k n)^{\eta} / \beta_n \right\} .$$

$$| I_{2,2,4} | = O(n^{-\frac{1}{2}} \lambda_n \beta_n^{-1}) \int_{1/n}^{\pi-(1/n)} w^{-3/2} | \phi(w) | \, dw = O \left\{ \lambda_n (\log^k n)^{\eta} / \beta_n \right\} .$$

Therefore

$$| I_{2,2} | = O \{ \lambda_n (\log^k n)^{\eta} / \beta_n \} .$$

and hence

$$| I_2 | = O \{ (\log^k n)^{\eta} / \beta_n \} . \tag{3.2}$$

Lastly, we consider

$$| I_3 | = O (n^{-1}) \int_{\pi-(1/n)}^{\pi} | \phi(w) | \sin w \, dw \sum_{\nu=2}^{n} \nu (\nu + 1) \lambda_\nu \beta_\nu^{-1}$$

$$= O (n^{-1}) \int_0^{1/n} w \, dw \, . \, O(n^3 \beta_n^{-1}) \sum_{\nu=2}^{n} \frac{\lambda_\nu}{\nu}$$

$$= O (\beta_n^{-1}) . \tag{3.3}$$

Combining (3.1), (3.2) and (3.3) , we obtain

$$| \tau_n | = O \{ (\log^k n)^{\eta} / \beta_n \} .$$

Therefore

$$\Sigma \mid \tau_n \mid /n = O \left\{ \sum \frac{1}{n \left[\prod_{\mu=1}^{k-1} log^{\mu} n \right] (log^{k} n)^{1+\epsilon-\eta}} \right\}$$

$$= O (1) .$$

This completes the proof of the theorem .

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