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HALL CURRENT EFFECT ON THERMAL INSTABILITY OF AN OLDROYDIAN VISCOELASTIC FLUID

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ABSTRACT

The thermal convection in a layer of electrically conducting Oldroydian viscoelastic fluid in the presence of Hall currents is studied . For the case of stationary convection, the Oldroydian fluid behaves like a Newtonian fluid and the magnetic field, Hall currents are found to have stabilizing and destabilizing effects respectively , The magnetic field and Hall currents introduce oscillatory modes in the system which were non-existent in their absence and when the strain retardation time be greater than the atress relaxation time .

1. INTRODUCTION

The problem of thermal convection in a horizontal layer of electrically conducting fluid in the presence of magnetic field has been discussed in detail by Chandrasekhar [3] . It is shown that a uniform magnetic field inhibits the onset of thermal convection . If an electric field is applied at right angles to the magnetic field, the whole current will not flow along the electric field . Hall current effect is this tendency of the electric current to flow across an electric field in the presence of a magnetic field . Gupta [5] has studied the problem of thermal convection in the presence of Hall current, The viscoelastic fluid described by the constitutive relations (1) (viz. eqs. (1) in section 2) is termed as an Oldroyd fluid ,

Eltayeb [4] has studied the convective instability in a rapidly rotating Oldroyd fluid . Sharma [7] has studied the stability of a layer of electrically conducting Oldroyd fluid (i. e. fluid obeying Oldroyd's constitutive equation) heated from below in the presence of magnetic field and has found the magnetic field to have a stabilizing influence , Sharma [8] has also studied the thermal instability of layer of Oldroyd fluid acted on by a uniform rotation and has found that the rotation has destabilizing as well as stabilizing effect under certain conditions in contrast to a Maxwell (viscoelastic) fluid where it has a destabilizing effect (Bhatia and Steiner [1]). An experimental demonstration by Toms and Strawbridge [9] reveals that a dilute solution of methyl methacrylate in *n*-butyl acetate agrees well with the theoretical model of Oldroyd fluid .

Keeping in mind the stability of some polymer solutions like a dilute solution of methyl methacrylate in *n*-butyl acetate, a re-consideration of the thermal convection in a layer of electrically conducting Oldroydian fluid in the presence of a uniform vertical magnetic field by including the Hall current is called for . This aspect forms the subject matter of the present paper .

2. Perturbation Equations

Consider an infinite horizontal layer of an electrically conducting Oldroydian fluid of depth d which is heated uniformly from below, acted on by a vertical magnetic field **H** (0, 0, H) and gravity force g (0, 0,$-g$) . The fluid is described by the constitutive relations

$$T_{ij} = - p\delta_{ij} + \tau_{ij} \; ;$$

$$\left(1+\lambda \frac{d}{dt}\right) \tau_{ij} = 2\mu \left(1+\lambda_0 \frac{d}{dt}\right) e_{ij} \, , \tag{1}$$

$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

where T_{ij}, τ_{ij}, e_{ij}, δ_{ij}, μ, λ and λ_0 denote respectively the stress tensor, shear stress tensor, rate-of-strain tensor, Kronecker delta, viscosity, stress relaxation time and strain retardation time . v_i, x_i, p and d/dt are velocity vector, position vector, isotropic pressure and mobile operator respectively . Relations of the type (1) were first proposed by Jeffreys for Earth and studied by Oldroyd [6]. Oldroyd [6] slso showed that many rheological equations of state, of general validity, reduce to (1) when linearized .

Let v (u, v, w) , ρ, T, μ_e, N and e stand respectively for velocity, density, temperature, magnetic permeability, number density and charge of an electron; ν, η, χ and α denote the kinematic viscosity, the electrical resistivity, the thermal diffusivity and the coefficient of thermal expansion respectively . The suffix zero refers to values at the reference level $z=0$. Then the equations of momentum, mass, energy and the Maxwell's equations are

$$\left(1+\lambda \frac{d}{dt}\right)\left[\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}.\nabla\right) \mathbf{v}\right] = \left(1+\lambda \frac{d}{dt}\right)\left[-\frac{1}{\rho_0}\nabla p +\right.$$

$$\left(1+\frac{\delta\rho}{\rho_0}\right)\mathbf{g}+\frac{\mu_e}{4\pi\rho_0}\left(\nabla\times\mathbf{H}\right)\times\mathbf{H}\Big]+\nu\left(1+\lambda_0\frac{d}{dt}\right)\nabla^2\mathbf{v}\,, \quad (2)$$

$$\nabla\cdot\mathbf{v}=0\,, \quad (3)$$

$$\frac{\partial T}{\partial t}+\left(\mathbf{v}\cdot\nabla\right)T=\chi\nabla^2 T\,, \quad (4)$$

$$\frac{\partial \mathbf{H}}{\partial t}=\left(\mathbf{H}\nabla\right)\mathbf{v}+\eta\nabla^2\mathbf{H}-\left(\frac{c}{4\pi Ne}\right)\nabla\times\left[\left(\nabla\times\mathbf{H}\right)\times\mathbf{H}\right], \quad (5)$$

$$\nabla\cdot\mathbf{H}=0, \quad (6)$$

where

$$\rho=\rho_0\left[1-\alpha\left(T-T_0\right)\right]. \quad (7)$$

In writing eq, (2), use has been made of the Boussinesq approximation . (7) represents the equation of state . The steady state solution is

$$\mathbf{v}=0\,,\ T=T_0-\beta z\,,\ \rho=\rho_0\left(1+\alpha\beta z\right), \quad (8)$$

where $\beta\left(=\left|\frac{dT}{dz}\right|=\frac{T_0-T_1}{d}\right)$ is the magnitude of uniform adverse temperature gradient and is positive , The change in density $\delta\rho$ caused by the perturbation θ in temperature is given by

$$\delta\rho=-\alpha\rho_0\,\theta\,. \quad (9)$$

On linearization, eqs. (2)—(6) give

$$\left(1+\lambda \frac{\partial}{\partial t}\right) \frac{\partial \mathbf{v}}{\partial t} = \left(1+\lambda \frac{\partial}{\partial t}\right)\left[-\frac{1}{\rho_0} \nabla \delta p + \frac{\mu_e}{4\pi\rho_0}\left(\nabla \times \mathbf{h}\right)\right.$$

$$\left. \times \mathbf{H} - \mathbf{g}\alpha\theta\right] + \nu\left(1 + \lambda_0 \frac{\partial}{\partial t}\right)\nabla^2 \mathbf{v} \,, \tag{10}$$

$$\nabla . \mathbf{v} = 0 \,, \quad \nabla . \mathbf{h} = 0, \tag{11}$$

$$\frac{\partial \theta}{\partial t} = \beta w + \chi \nabla^2 \theta \,, \tag{12}$$

$$\frac{\partial \mathbf{h}}{\partial t} = \left(\mathbf{H} \cdot \nabla\right) \mathbf{v} + \eta \nabla^2 \mathbf{h} - \left(\frac{c}{4\pi N e}\right) \nabla \times \left[\left(\nabla \times \mathbf{h}\right) \times \mathbf{H}\right]. \tag{13}$$

Equations (10) - (13) give

$$\left(1 + \lambda \frac{\partial}{\partial t}\right)\left[\frac{\partial}{\partial t} \nabla^2 w - g\alpha\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) - \frac{\mu_e H}{4\pi\rho_0} \frac{\partial}{\partial z} \nabla^2 h_z\right]$$

$$= \nu \left(1 + \lambda_0 \frac{\partial}{dt}\right) \nabla^4 w \,, \tag{14}$$

$$\left(1+\lambda \frac{\partial}{\partial t}\right) \frac{\partial \zeta}{\partial t} - \nu \left(1+\lambda_0 \frac{\partial}{\partial t}\right) \nabla^2 \zeta = \frac{\mu_e H}{4\pi\rho_0} \left(1+\lambda \frac{\partial}{\partial t}\right) \frac{\partial \xi}{\partial z} \,, \tag{15}$$

$$\left(\frac{\partial}{\partial t} - \chi \nabla^2\right) \theta = \beta w \,, \tag{16}$$

$$\left(\frac{\partial}{\partial t} - \eta \nabla^2\right) \xi = H \frac{\partial \zeta}{\partial z} + \frac{cH}{4\pi N e} \nabla^2 \frac{\partial hz}{\partial z} \,, \tag{17}$$

$$\left(\frac{\partial}{\partial t} - \eta \nabla^2\right) h_z = H \frac{\partial w}{\partial z} - \frac{cH}{4\pi N e} \frac{\partial \xi}{\partial z} \,. \tag{18}$$

Here the boundaries are taken to be free as well as perfect conductors of heat and the adjoining medium is taken to be electrically nonconducting. Though the case of two free boundaries is artificial except in the case of stellar atmospheres, it allows us to obtain the analytical solution without affecting the essential features of the problem. The boundary conditions in this case are

$$w = \frac{\partial w^2}{\partial z^2} = \theta = 0 \,, \quad \frac{\partial \zeta}{\partial z} = 0 \,, \; \xi = 0 \quad \text{at } z = 0 \text{ and } z = d \,, \tag{19}$$

and **h** are continuous with an external vacuum field on a nonconducting boundary.

3. Dispersion Relation

We now analyze the disturbances into normal modes, assume that the perturbations w, θ, h_z, ζ and ξ have the forms

$$[\, w,\ \theta,\ h_z,\ \zeta,\ \xi \,] = [\, W(z),\ \otimes(z),\ K(z),\ Z(z),\ X(z) \,]\ exp\,(ik_x x + ik_y y + nt)\,, \tag{20}$$

where n is the growth rate, $k = (\, k_x^2 + k_y^2 \,)^{\frac{1}{2}}$ and k_x, k_y are wave numbers along the x and y directions respectively.

Putting the coordinates x, y, z in the new unit of length d and letting $a = kd$, $\sigma = nd^2/\nu$, $F = \lambda\nu/d^2$, $\Gamma = \lambda_0\nu/d^2$, $p_1 = \nu/\chi$, $p_2 = \nu/\eta$ and $D = d/dz$, the non-dimensional forms of eqs. (14)-(18), using expression (20), are

$$(\, 1 + F\sigma \,) \left[\sigma\, (\, D^2 - a^2 \,)\, W + \left(\frac{g\alpha d^2}{\nu} \right) a^2 \otimes - \frac{\mu_e H d}{4\pi\rho_0\nu} (D^2 - a^2)\, DK \right]$$

$$= (\, 1 + \Gamma\sigma \,)\, (D^2 - a^2\,)^2\, W \,, \tag{21}$$

$$\left[[1+F\sigma) \sigma - (1+\Gamma\sigma) (D^2 - a^2) \right] Z = \frac{\mu_e H d}{4\pi\rho_0 \nu} (1+F\sigma) DX , \quad (22)$$

$$(D^2 - a^2 - p_1 \sigma) \otimes = - \left(\frac{\beta d^2}{\chi} \right) W , \quad (23)$$

$$(D^2 - a^2 - p_2 \sigma) X = - \left(\frac{Hd}{\eta} \right) DZ - \frac{cH}{4\pi N e \eta d} (D^2 - a^2) DK , \quad (24)$$

$$(D^2 - a^2 - p_2 \sigma) K = - \left(\frac{Hd}{\eta} \right) DW + \left(\frac{cHd}{2\pi N e \eta} \right) DX , \quad (25)$$

where $D = d/dz$. Eliminating Z, X, K and $\otimes$ from eqs . (21) — (25), we obtain

$$[\{\sigma(1+F\sigma)-(1+\Gamma\sigma) (D^2-a^2)\} \{(D^2 -a^2-p_2\sigma)^2 + MD^2 (D^2-a^2)\}$$

$$+ QD^2 (D^2-a^2-p_2\sigma) (1+F\sigma)] [(D^2-a^2) (D^2-a^2-p_1\sigma)$$

$$- (1+ F\sigma) Ra^2] W + Q (1+ F\sigma) (D^2-a^2) (D^2-a^2-p_1\sigma)$$

$$[\{(1+F\sigma) \sigma - (1+\Gamma\sigma) (D^2-a^2)\} (D^2-a^2-p_2\sigma)$$

$$+ QD^2 (1+F) D^2 W = 0 , \quad (26)$$

where $Q = \frac{\mu_e H^2 d^2}{4\pi\rho\nu\eta}$ is the Chandrasekhar number , $R = \frac{g\alpha\beta d^4}{\nu\chi}$ is the Rayleigh number and $M=\left(\frac{cH}{4\pi N e \eta} \right)^2$ is a non-dimensional number accounting for the Hall effects .

The boundary conditions (19) transform to

$$\left.\begin{array}{l} W = D^2W = \otimes = DZ = X = 0 \text{ at } z = 0 \text{ and } z = 1 \\ \text{and } \mathbf{h} \text{ is continuous with an external vaccum field .} \end{array}\right\} \quad (27)$$

Using the above boundary conditions, it can be shown that all the even order derivatives of W must vanish for $z = 0$ and 1 and hence the proper solution of (26) characterizing the lowest mode is

$$W = W_0 \sin \pi z , \quad (28)$$

where W_0 is a constant. Substituting (28) in eq. (26), we obtain the dispersion relation

$$R_1 x = \frac{(1+x)(1+x+ip_1\sigma_1)}{(1+i\sigma_1\pi^2 F)} \left[(1+x)(1+i\sigma_1\pi^2\Gamma) + i\sigma_1 (1+i\sigma_1\pi^2 F) \right]$$

$$+ \frac{Q_1(1+x)(1+x+ip_1\sigma_1)[\{(1+x)(1+i\sigma_1\pi^2\Gamma)+i\sigma_1(1+i\sigma_1\pi^2 F)\}}{[\{(1+x)(1+i\sigma_1\pi^2\Gamma)+i\sigma_1(1+i\sigma_1\pi^2 F)\}}$$

$$\times \frac{(1+x+ip_2\sigma_1) + Q_1(1+i\sigma_1\pi^2 F)]}{\{(1+x+ip_2\sigma_1)^2 + M(1+x)\} + Q_1(1+i\sigma_1\pi^2 F)(1+x+ip_2\sigma_1)]} \quad (29)$$

where $x = a^2/\pi^2$, $R_1 = R/\pi^4$, $Q_1 = Q/\pi^2$ and $i\sigma_1 = \sigma/\pi^2$.

In the limit of vanishing M, the dispersion relation (29) reduces to the result (Sharma [7]). In the limit of vanishing M and Γ, the dispersin relation also reduces to the case of a Maxwell fluid (Bhatia and Steiner [2]).

4. The Stationary Convection

For the case of stationary convection *i. e.* $\sigma = 0$, eq. (29) reduces to

$$R_1 = \left(\frac{1+x}{x}\right)\left[(1+x)^2 + \frac{Q_1\{(1+x)^2 + Q_1\}}{(1+x)(1+x+M)+Q_1}\right], \tag{30}$$

the result obtained by Gupta [5] .

We thus find that for the stationary convection, the stress relaxation time parameter F and the strain retardation time parameter Γ vanish with σ and the Oldroydian viscoelastic fluid behaves like a Newtonian viscous fluid . To fiud the effect of magnetic field and Hall current on the system, we examine the natures of dR_1/dQ_1 and dR_1/dM analytically .

From eq. (30), it follows that

$$\frac{dR_1}{dQ_1} = \frac{(1+x)^2}{x}\cdot\frac{\{(1+x)^2 + 2Q_1\}(1+x+M)+Q_1}{[(1+x)(1+x+M)+Q_1]^2}, \tag{31}$$

which is positive and so the magnetic field has a stabilizing effect for stationary convection .

Equation (30) also yields

$$\frac{dR_1}{dM} = -\frac{Q_1(1+x)^2\{(1+x)^2 + Q_1\}}{x[(1+x)(1+x+M)+Q_1]^2}, \tag{32}$$

which is always negative . The Hall current therefore has a destabilizing effect on the system for stationary convection .

5. Stability of the System and Oscillatory Modes

Multiplying eq. (21) by W^* , the complex conjugate of W, integrating over the range of z and using (22)-(25), we obtain

$$\sigma I_1 + \frac{(1+\Gamma\sigma)}{(1+F\sigma)} I_2 - \frac{g\alpha\chi a^2}{\nu\beta}(I_3 + \sigma^* p_1 I_4) + \frac{\mu_e\eta}{4\pi\rho\nu}(I_5 + \sigma^* p_2 I_6)$$

$$+\frac{\mu_e \eta d^2}{4\pi\rho\nu}\left(I_7+\sigma p_2 I_8\right)+d^2\left[\sigma^* I_9+\frac{1+\Gamma\sigma^*}{1+F\sigma^*} I_{10}\right]=0\ , \tag{33}$$

and $I_1 - I_{10}$ are all positive definite and are given in the appendix , Putting $\sigma = \sigma_r + i\,\sigma_i$ in eq . (33) and separating the real and imaginary parts , we obtain

$$\sigma_r\left[I_1+\frac{(F+\Gamma)(I_2+d^2 I_{10})}{(1+\sigma_r F)^2+\sigma_i^2 F^2}+\frac{\mu_e\eta}{4\pi\rho\nu}p_2\left(I_6+d^2 I_8\right)\right.$$

$$\left.+d^2 I_9-\frac{g\alpha\chi a^2}{\nu\beta}p_1 I_4\right]$$

$$=-\left[\frac{(I_2+d^2 I_{10})\{1+F\Gamma(\sigma_r^2+\sigma_i^2)\}}{(1+\sigma_r F)^2+\sigma_i^2 F^2}\right.$$

$$\left.+\frac{\mu_e\eta}{4\pi\rho\nu}\left(I_5+d^2 I_7\right)-\frac{g\alpha\chi a^2}{\nu\beta}I_3\right], \tag{34}$$

and

$$\sigma_i\left[I_1+\frac{(\Gamma-F)(I_2-d^2 I_{10})}{(1+\sigma_\gamma F)^2+\sigma_i^2 F^2}+\frac{\mu_e\eta d^2}{4\pi\rho\nu}b_2 I_8+\frac{g\alpha\chi a^2}{\nu\beta}p_1 I_4\right.$$

$$\left.-\frac{\mu_e\eta}{4\pi\rho\nu}p_2 I_6-d^2 I_9\right]=0\ . \tag{35}$$

It is inferred from eq (34) that σ_r may be positive or negative meaning thereby that the systèm may be stable or unstable . Equation (35) implies that in the absence of magnetic field (and hence Hall currents), the oscillatory modes are not allowed if the strain retardation time λ_0 is greater than the stress relaxation time λ $\left[\text{for } \Gamma \left(= \frac{\lambda_0 \nu}{d^2}\right) > F\left(= \frac{\lambda \nu}{d^2}\right)\right]$. The magnetic field and Hall currents, thus, introduce oscillatory modes in the system which were non-existent in their absence and when the strain retardation time be greater than the stress relaxation time.

6, **Results**

The conclusions of the paper are presented here.

(i) The Oldroyd viscoelastic fluid behaves like a Newtonian fluid for the case of stationary convection. For stationaro convection, the magnetic field and Hall currents are found to have stabilizing and destabilizing effeets respectively.

(ii) The magnetic field introduces oscillatory modes in the system which were non-existent in its absence and when the strain retardation time be greater than the stress relaxation time.

(iii) Hall currents introduce oscillatory modes in the system which were not present in the absence of Hall currents and when the strain retardation time be greater than the stress relaxation time.

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REFERENCES

[1] P. K. Bhatia and J. M. Steiner, Convective instability in a rotating viscoelastic fluid layer, *Zeitschr Angew. Math. Mech.* 52 (1972), 321-327.

[2] P. K. Bhatia and J. M. Steiner, Thermal instability in a visco-elastic fluid layer in hydromagnetics, *J. Math. Anal. Appl.* **41** (1973), 271-283,

[3] S. Chandrasekhar, *Hydrodynomic and Hydromagnetic Stability*. Clarendon Press, Oxford, 1961.

[4] I. A. Eltayeb, Convective instability in a rapidly rotating viscoelastic layer, *Zeitschr. Angew. Math. Mech.* **55** (1975), 599-604.

[5] A. S. Gupta, Hall effects on thermal instability, *Rev. Roumaine Math. Pures Appl.* **12** (1967), 665-677.

[6] J. G. Oldroyd, Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids, *Proc, Roy. Soc. London Ser. A.* A **245** (1958), 278-297.

[7] R. C. Sharma, Thermal instability in a viscoelastic fluid in hydromagnetics, *Acta Phys. Hungar*. 38 (1975), 293-298.

[8] R. C. Sharma, Effect of rotation on thermal instability of a viscoelastic fluid, *Acta Phys. Hungar*, 40 (1976), 11-17,

[9] B. A. Toms and D. J. Strawbridge, Elastic and viscous properties of dilute solutions of polymethyl methacrylate in organic liquids, *Trans. Faraday Soc.* 49 (1953), 1225-1232.

Appendix

The integrals $I_1 - I_{10}$ in eq. (33) are given as under :

$$I_1 = \int_0^1 (\,|DW|^2 + a^2 |W|^2)\, dz\,,$$

$$I_2 = \int_0^1 (\,|D^2W|^2 + 2a^2 |DW|^2 + a^4 |W|^2)\, dz\,,$$

$$I_3 = \int_0^1 (\,|D\otimes|^2 + a^2 |\otimes|^2)\, dz\,,$$

$$I_4 = \int_0^1 |\otimes|^2\, dz\,,$$

$$I_5 = \int_0^1 (\,|D^2K|^2 + 2\,a^2 |DK|^2 + a^4 |K|^2)\, dz\,,$$

$$I_6 = \int_0^1 (\,|DK|^2 + a^2 |K|^2)\, dz\,,$$

$$I_7 = \int_0^1 (\,|DX|^2 + a^2 |X|^2)\, dz\,,$$

$$I_8 = \int_0^1 |X|^2\, dz\,,$$

$$I_9 = \int_0^1 |z|^2\, dz\,,$$

$$I_{10} = \int_0^1 (\,|Dz|^2 + a^2 |z|^2)\, dz\,.$$