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ON THE BESSEL TRANSFORMS

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ABSTRACT

In this paper, a new variant of the Hankel transform is introduced. The corresponding inversion theorem is established and some operational formulas are obtained. These, results are then used in solving certain partial differential equations. Finally, we show that this transform is connected with the Hankel-Bessel-Schwartz transform by means of an interesting Parseval type relation.

1. INTRODUCTION

A. L. Schwartz [8] first established the inversion theorem of the following modified Hankel transformation

$$F_1(y) = (h_{1,\mu} f)(y) = \int_0^\infty x^{2\mu+1} b_\mu(xy) f(x)\, dx , \tag{1}$$

where $b_\mu(z) = z^{-\mu} J_\mu(z)$, $J_\mu(z)$ being the Bessel function of the first kind of order μ. This transform has been investigated in distributional spaces by L. S. Dube and J. N. Pandey [2], W. Y. Lee [4], J. M. Méndez [6], and also by G. Altenburg [1] who calls it the Bessel transformation.

In this paper the following new variant of the Hankel transform

$$G_2(y) = (h_{2,\mu}\, g)(y) = y^{2\mu+1}\int_0^\infty b_\mu(xy)\, g(x)\, dx, \tag{2}$$

connected closely with (1), is considered and its inversion formula is given. Some important operational formulas are obtained, which will be used in solving a certain class of partial differential equations. Finally, we prove that both transforms (1) and (2) are related by means of Parseval's equations. These Parseval relations suggest new definitions of the transforms $h_{1,\mu}$ and $h_{2,\mu}$ on adequate spaces of generalized functions.

2. Preliminary results

The function $b_\mu(x)$ is a solution of the differential equation [1, p. 213]

$$y'' + (2\mu+1)x^{-1}y' + y = 0,$$

and, therefore, satisfies

$$x^2 b_{\mu+2}(x) - 2(\mu+1)\, b_{\mu+1}(x) + b_\mu(x) = 0, \tag{3}$$

In this work we shall need the formulas [3, p. 16]

$$D_y\, [\, y^{2\mu}\, b_\mu(uy)\,] = y^{2\mu-1}\, b_{\mu-1}(uy) \tag{4}$$

$$D_y\, b_\mu(uy) = -\, yu^2\, b_{\mu+1}(uy). \tag{5}$$

An integration by parts and the use of (4) and (5) yield

$$\int_0^\lambda u^{2\mu+1}\, b_\mu(xu)\, b_\mu(yu)\, du =$$

$$= \frac{\lambda^{2\mu+2}}{x^2-y^2}\left[\, x^2\, b_{\mu+1}(\lambda x)\, b_\mu(\lambda y) - y^2 b_\mu(\lambda x)\, b_{\mu+1}(\lambda y)\,\right], \tag{6}$$

provided that $\mu \geqslant -\frac{1}{2}$.

From the asymptotic expansion of the Bessel functions [7,p. 122] we find that

$$b_\mu(x) = \sqrt{\left(\frac{2}{\pi}\right)}\, x^{-\mu-1/2} \cos\left(x - \frac{\mu\pi}{2} - \frac{\pi}{4}\right), \text{ as } x \to \infty \tag{7}$$

We also require the fact that [7, p. 23]

$$b_\mu(x) = O(1), \quad \text{as } x \to 0+ \tag{8}$$

Recall finally the value of the integral, $\mu > -1$ [10, p. 406]

$$\int_0^\infty t^{2\mu+1}\, b_{\mu+1}(at)\, b_\mu(ct)\, dt = \begin{cases} a^{-2\mu-2}, & a > c \\ \frac{1}{2} c^{-2\mu-2}, & a = c \\ 0, & a < c \end{cases} \tag{9}$$

3. Inversion formula.

THEOREM 1. *Let* $\mu \geqslant -\frac{1}{2}$. *If* $f(y)$ *is an integrable function such that*

$$\int_0^\infty y^{-\mu-1/2} | f(y) | \, dy < \infty,$$

then

$$\lim_{\lambda \to \infty} x^{2\mu+1} \int_0^\lambda u^{2\mu+1}\, b_\mu(xu) \int_0^\infty b_\mu(yu) f(y) dy\, du = \frac{1}{2}\left[f(x+0) + f(x-0)\right],$$

in a neighborhood of every point $y = x$, *where* $f(y)$ *is of bounded variation.*

Proof. The following proof, just as Schwartz's proof [8], is essentially that given by Titchmarsh [9] in his verification of the inversion theorem for the Hankel transform. Thus, suppose that x is fixed in $0 < x < \infty$. Through an argument similar to the one employed in [8], it can be deduced that

$$\lim_{\lambda\to\infty} x^{2\mu+1} \int_0^{\lambda} u^{2\mu+1} b_\mu(xu) \int_0^{x-\delta} b_\mu(yu) f(y)\, dy\, du = 0, \qquad (10)$$

δ being a small positive number. However, we are interested in emphasizing the next result

$$\lim_{\lambda\to\infty} x^{2\mu+1} \int_0^{\lambda} u^{2\mu+1} b_\mu(xu) \int_{x+\delta}^{\infty} b_\mu(yu) f(y)\, dy\, du = 0. \qquad (11)$$

Indeed, since $| z^{\mu+\frac{1}{2}} b_\mu(z) | = O(1)$ in $0 < z < \infty$, it follows that

$$| u^{\mu+\frac{1}{2}} \int_{x+\delta}^{\infty} b_\mu(yu) f(y)\, dy | = O\left(\int_0^{\infty} y^{-\mu-1/2} | f(y) |\, dy \right).$$

Then, we may invoke Fubini's theorem to obtain, in view of (6),

$$x^{2\mu+1} \int_0^{\lambda} u^{2\mu+1} b_\mu(xu) \int_{x+\delta}^{\infty} b_\mu(yu) f(y)\, dy\, du$$

$$= x^{2\mu+3} \lambda^{2\mu+2} b_{\mu+1}(\lambda x) \int_{x+\delta}^{\infty} b_\mu(\lambda y)(x^2-y^2)^{-1} f(y)\, dy \qquad (12)$$

$$- x^{2\mu+1} \lambda^{2\mu+2} b_\mu(\lambda x) \int_{x+\delta}^{\infty} y^2 b_{\mu+1}(\lambda y)(x^2-y^2)^{-1} f(y)\, dy.$$

Note that the first term of the right-hand side of (12) can be written as

$$x^{\mu+\frac{3}{2}} (\lambda x)^{\mu+\frac{3}{2}} b_{\mu+1}(\lambda x) \int_{x+\delta}^{\infty} (\lambda y)^{\mu+1/2} b_\mu(\lambda y)(x^2-y^2)^{-1} y^{-\mu-1/2} f(y)\, dy. \qquad (13)$$

The factor outside of the integral sign is always bounded no matter what the value of λ may be, by virtue of (7) and (8). On the other hand, for very large values of λ, the integral in (13) reduces to

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_{x+\delta}^{\infty} \cos\left(\lambda y - \frac{\mu\pi}{2} - \frac{\pi}{4}\right)(x^2-y^2)^{-1} y^{-\mu-\frac{1}{2}} f(y)\, dy$$

which tends to zero as $\lambda\to\infty$, because of the Riemann - Lebesgue theorem. Hence, (13) vanishes when $\lambda\to\infty$. The same argument may be applied to the second term of (12) . Thus we conclude that (11) holds .

Since $f(y)$ is of bounded variation in a neighborhood of $y=x$, we may put

$$y^{-2\mu-1} f(y) = x^{-2\mu-1} f(x+0) + \Psi_1(y) - \Psi_2(y),$$

on $(x, x+\delta)$, where Ψ_2 and Ψ_2 are positive and increasing functions such that $\Psi_1(x+0) = \Psi_2(x+0) = 0$. Then, by applying the second mean-value theorem, we arrive at

$$x^{2\mu+1}\int_0^{\lambda} b_\mu(xu)\, u^{2\mu+1}\int_x^{x+\delta} b_\mu(yu) f(y)\, dy\, du$$

$$= f(x+0)\int_0^{\lambda} b_\mu(xu\ \ u^{2\mu+1}\int_x^{x+\delta} y^{2\mu+1} b_\mu(yu)\, dy\, du$$

$$+\Psi_1(x+\delta)\int_0^{\lambda} u^{2\mu+1} b_\mu(xu)\int_\xi^{x+\delta} y^{2\mu+1} b_\mu(yu)\, dy\, du \qquad (14)$$

$$-\Psi_2(x+\delta)\int_0^{\lambda} u^{2\mu+1} b_\mu(xu)\int_{\xi'}^{x+\delta} y^{2\mu+1} b_\mu(yu)\, dy\, du,$$

with $x<\xi, \zeta'<x+\delta$. The first integral of the right-hand side of (14) tends to $\frac{1}{2} f(x+0)$ when $\lambda\to\infty$, as it follows from (4) and (9), where as the remaining terms are $O(\varepsilon)$, since $\Psi_1(y)\leqslant\varepsilon$ and $\Psi_2(y)\leqslant\varepsilon$ for

all y, $x \leqslant y \leqslant x+\delta$, on choosing δ sufficiently small. Hence we have seen that

$$\lim_{\lambda\to\infty} x^{2\mu+1} \int_0^\lambda u^{2\mu+1} b_\mu(xu) \int_x^{x+\delta} b_\mu(yu) f(y)\, dy\, du = \tfrac{1}{2} f(x+0) \tag{15}$$

Next, we can derive in a similar way

$$\lim_{\lambda\to\infty} x^{2\mu+1} \int_0^\lambda u^{2\mu+1} b_\mu(xu) \int_{x-\delta}^{x} b_\mu(yu) f(y)\, dy\, du = \tfrac{1}{2} f(x-0) \tag{16}$$

The proof of the theorem is now completed cy combining (10), (11), (15) and (16).

Remark 1. Schwartz [8] proves the inversion theorem for the transform (1) by assuming that $x^{2\mu+1} f(x)$ is absolutely integrable on $0<x<\infty$ and that

$$\int_0^1 x^{\mu+1/2} \mid f(x) \mid dx$$

exists. These hypotheses can be simplified. Indeed, the absolute integrability of $x^{\mu+1/2} f(x)$ on $0<x<\infty$ is enough to guarantee the validity of the inversion formula for the transform (1). This point can be verified by proceeding as in the proof of Theorm 1.

4. An operational calculus and its applications

In the sequel, we assume that $\mu \geqslant -\frac{1}{2}$. Let $f(x) \in C^1(0,\infty)$ such that $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x\to\infty} x^{-\mu-1/2} f(x) = 0$. Then, upon integrating by parts and making use of (5), we get

$$y^{2\mu+1} \int_0^\infty b_\mu(xy) f'(x)\, dx$$

$$= y^{2\mu+1} \left\{ \Big[b_\mu(xy)\ f(x) \Big]_{x\to 0}^{x\to\infty} + y^2 \int_0^\infty b_{\mu+1}(xy)\ x f(x)\, dx \right\}.$$

Hence

$$h_{2,\mu}[f'(x)] = h_{2,\mu+1}[f(x)] \tag{17}$$

If $f(x) \in C^2(0,\infty)$, and *if* $f^{(i)}(x)$ tends to zero as $x \to 0^+$, whereas $x^{-\mu-1/2} f^{(i)}(x)$ vanishes when $x \to \infty$ ($i=0, 1$), by using (17) and the formula for integrating by parts, we obtain

$$h_{2,\mu}[f''(x)] = h_{2,\mu+1}[xf'(x)]$$

$$= y^{2\mu+3} \int_0^\infty f(x) \{x^2 y^2 b_{\mu+2}(xy) - b_{\mu+1}(xy)\} dx .$$

Finally, from (3) we have

$$h_{2,\mu}[f''(x)] = (2\mu+1) h_{2,\mu+1}[f(x)] - y^2 h_{2,\mu}[f(x)] \tag{18}$$

Now, in the above assumptions, it is deduced that

$$h_{2,\mu}[D^2 f(x) - (2\mu+1) D(x^{-1} f(x))] = -y^2 h_{2,\mu}[f(x)], \tag{19}$$

as a simple consequence of (17) and (18).

This formula can be fruitfully utilized in solving a certain class of partial differential equations, We illustrate it with an example in which we find the solution $v = v(r,t)$ of the equation

$$\frac{\partial^2 u}{\partial r^2} - \frac{2\mu+1}{r}\frac{\partial v}{\partial r} + \frac{2\mu+1}{r^2} v + \frac{\partial^2 v}{\partial t^2} = 0, \quad \mu \geqslant 0, \tag{20}$$

satisfying the conditons

$v(r,t)$ and $\frac{\partial v(r;t)}{\partial r}$ vanish *as* $r \to 0^+$

$v(r,t) = O(r^{\mu+1/2})$ and $\frac{\partial v(r,t)}{\partial r} = O(r^{\mu+1/2})$, *as* $r \to \infty$

$v(r,t) \to 0$, *as* $t \to \infty$

$v(r,0) = f(r)$

Set $V(\rho,t) = h_{2,\mu}[u(r,t)]$. By applying $h_{2,\mu}$ to (20) and invoking formula (19), we convert (20) into

$$\frac{\partial^2 V}{\partial t^2} - \rho^2 V = 0 \quad , \quad 0<\rho<\infty$$

$$V(\rho,t) \to 0 \ , \ as\ t \to \infty$$

$$V(\rho,0) = F(\rho) = h_{2,\mu}[f(r)] \ ,$$

whose solution is

$$V(\rho,t) = F(\rho)\, e^{-\rho t}$$

Inverting this result , we obtain

$$v(r,t) = r^{2\mu+1}\int_0^{\infty} b_\mu(r\rho)\, F(\rho)\, e^{-\rho t}\, d\rho$$

as our formal solution of (20) .

5. Parseval relations for Bessel transforms

$L(0,\infty)$ denotes the space of all functions $f(x)$ that are Lebesgue integrable on $0<x<\infty$, We now establish

THEOREM 2. *Let* $x^{\mu+1/2} f(x)$ *and* $y^{\mu+1/2} G_1(Y)$ *belong to* $L(0,\infty)$. *Assume that* $F(Y) = h_{1,\mu}[f(x)]$ *and* $g(x) = h_{1,\mu}^{-1}[G_1(y)]$ $= h_{1,\mu}[G_1(Y)]$, *where* $\mu \geqslant -\frac{1}{2}$. *Then*

$$\int_0^{\infty} x^{2\mu+1} f(x)\, g(x)\, dx = \int_0^{\infty} y^{2\mu+1} F_1(Y)\, G_1(Y)\, dy \tag{21}$$

Proof. In fact, by the absolute convergence of the y-integral , we may invert the order of integraion to obtain (21) .

Similarly, for the second Bessel transform . we have

THEOREM 3. *Suppose* $x^{-\mu-1/2} f(x)$ *and* $y^{-\mu-1/2} G_2(y)$ *belong to* $L(0,\infty)$, *where* $\mu \geqslant -\frac{1}{2}$. *If* $F_2(y) = h_{2,\mu}[f(x)]$ *and* $g(x) = h_{2,\mu}^{-1}[G_2(Y)]$ $= h_{2,\mu}[G_2(Y)]$, *then*

$$\int_0^\infty x^{-2\mu-1} f(x) g(x)\, dx = \int_0^\infty Y^{-2\mu-1} F_2(Y)\, G_2(Y)\, dy$$

We end this section by establishing a curious Parseval type relation, which involves both the Bessel transforms $h_{1,\mu}$ and $h_{2,\mu}$.

THEOREM 4 . *Let* $x^{\mu+1/2} f(x)$ *and* $Y^{-\mu-1/2}\, G_2(y)$ *belong to* $L(0,\infty)$, *where* $\mu \geqslant -\frac{1}{2}$, *If* $F_1(y) = h_{1,\mu}[f(x)]$ *and* $g(x) = h^{-1}_{2,\mu}[G_2(y)]$ $= h_{2,\mu}[G_2(y)]$,

then $$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_1(y)\, G_2(y)\, dy ,$$

that is , $$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty h_{1,\mu}[f(x)] \cdot h_{2,\mu}[g(x)]\, dy \tag{22}$$

Proof . The absolute convergence of the y-integral allows us to apply once more the Fubini s theorem to get

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty f(x) \left(x^{2\mu+1} \int_0^\infty b_\mu(x)\, G_2(y)\, dy \right) dx =$$

$$= \int_0^\infty G_2(y) \int_0^\infty x^{2\mu+1}\, b_\mu(xy)\, f(x)\, dx\, dy = \int_0^\infty F_1(y)\, G_2(y)\, dy .$$

Remark 2 . Note that the last Parseval relation does not involve any weight function, in contrast to the Parseval expressions of Theorems 2 and 3 . In fact, Zemanian s method for extending the Hankel transformation to a space of generalized functions of slow growth is based upon the existence of a Parseval relation analogous to (22) [11, *Th* . 5. 1-2, *p*. 127] Thus, the generalized Hankel transformation is defined as an extension of the mentioned Parseval equation [11,*p*. 142] . Hence, expression (22) suggests a new definition of the generalized Bessel transformations $h_{1,\mu}$ and $h_{2,\mu}$, quite different from those considered in [1], [2], [4], and [6] .

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