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ON THE MULTIDIMENSIOAL LAPLACE TRANSFORM AND ITS APPLICATIONS

By

RAJNI GUPTA

Department of Mathematics, University of Rajasthan,

Jaipur - 302004, India

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ABSTRACT

In this paper, first we establish an interesting theorem for the multidimensional Laplace transform. Next we obtain a multiple infinite integral involving the product of H-functions of one and several variables having different arguments with the help of our theorem. The main theorem and the integral are quite general in nature and yield a number of (known or new) results as particular cases.

1. INTRODUCTION AND NOTATIONS

The multidimensional Laplace transform is defined by the following integral equation [1] :

$$g(s_1, \ldots, s_r) = (s_1 \ldots s_r) \int_0^\infty \cdots \int_0^\infty exp\left\{ - \sum_{i=1}^{r} s_i x_i \right\}$$

$$f(x_1, \ldots, x_r)\, dx_1 \ldots dx_r \qquad \ldots (1.1)$$

where Re $(s_i) > 0$, $\forall i \in \{1, \ldots, r\}$ and the function $f(x_1, \ldots, x_r)$ is so chosen that the above integral is absolutely convergent. As usual, we shall represent (1.1) by the following notation :

$$g(s_1,\ldots,s_r) = L\{f(x_1,\ldots,x_r);\, s_1,\ldots,s_r\} \qquad \ldots (1.2)$$

The analogue of the well-known Parseval–Goldstein theorem (concerning the Laplace transform) for the transform (1.1) is as follows:

If

$$g_1(s_1,\ldots,s_r) = L\{f_1(x_1,\ldots,x_r);\, s_1,\ldots,s_r\}$$

and

$$g_2(s_1,\ldots,s_r) = L\{f_2(x_1,\ldots,x_r);\, s_1,\ldots,s_r\},$$

then

$$\int_0^\infty \cdots \int_0^\infty g_1(x_1,\ldots,x_r)\, f_2(x_1,\ldots,x_r)\, \frac{dx_1}{x_1} \cdots \frac{dx_r}{x_r}$$

$$= \int_0^\infty \cdots \int_0^\infty g_2(x_1,\ldots,x_r)\, f_1(x_1,\ldots,x_r)\, \frac{dx_1}{x_1} \cdots \frac{dx_r}{x_r} \qquad (1.3)$$

provided that the integrals involved exisi .

The H-function of several variables occurring in this peper is a speeial case of ihe general H-function of several variables introduced and studied earlier by Srivastava and Panda (cf. [7] and [8]) .

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The parameters of this function will be displayed in the famiiar contracted notation ([6] , *p.* 251 , *Eq.* (*C*.1)) . The details of this function can be found in the papers and the book referred to above .

2. MAIN THEOREM

Under the hypo hesis (1.2) , *let*

$$h(s_1,\ldots,s_r) = L\left\{\left(\prod_{i=1}^{r}\left(x_i^{-\rho_i}\right)\right) H_1\left[y_1 x_1^{-\omega_1},\ldots,y_r x_r^{-\omega_r}\right] f\left(x_1,\ldots,x_r\right);\; s_1,\ldots,s_r\right\}. \qquad (2.1)$$

Then

$$h(s_1,\ldots,s_r)=(s_1\ldots s_r)\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{r}\left\{x_i^{\rho_i-1}(x_i+s_i)^{-1}\right\}$$

$$g(x_1+s_1,\ldots,x_r+s_r)$$

$$\cdot H^{O,O:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1+1;\ldots;p_r,q_r+1}\left[\begin{matrix}y_1x_1^{\omega_1}\\ \vdots\\ y_rx_r^{w_r}\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j',\ldots;\alpha_j^{(r)})_{1,p}:\\ (b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,q}:\end{matrix}\right.$$

$$\left.\begin{matrix}(c_j',\gamma_j')_{1,p_1};\ \ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\ (d_j',\delta_j')_{1,q_1},(1-\rho_1,\omega_1);\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r},(1-\rho_r,\omega_r)\end{matrix}\right]$$

$$dx_1\ldots dx_r \qquad \ldots (2.2)$$

provided that

(i) *the n-dimensional Laplace transforms of* $|g(x_1,\ldots,x_r)|$ *and*

$$\left|x_1^{-\rho_1}\ldots x_r^{-\rho_r}\,g(x_1,\ldots,x_r)\,H_1\left[y_1x_1^{-\omega_1},\ldots,y_rx_r^{-\omega_r}\right]\right| \text{ exist};$$

(ii) $U_i-\omega_i>0$, $|\arg y_i|<\frac{1}{2}(U_i-\omega_i)\pi$,

where

$$U_i=-\sum_{j=1}^{p}\alpha_j^{(i)}-\sum_{j=1}^{q}\beta_j^{(i)}+\sum_{j=1}^{n_i}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}\gamma_j^{(i)}$$

$$+\sum_{j=1}^{m_i}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}\delta_j^{(i)},\ \forall\, i\in\{1,\ldots,r\};$$

(iii) $\mathrm{Re}(s_i)>0$, $\omega_i>0$, $\mathrm{Re}(\rho_i)+\omega_i\min\limits_{1\leqslant j\leqslant m_i}$

$$[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]>0,\ \forall i\in\{1,\ldots,r\};$$

(iv) *The multiple integral on the right-hand side of (2.2) is absolutely convergent.*

Proof. We have

$$(s_1 \ldots s_r)\ (s_1 + a_1)^{-1} \ldots (s_r + a_r)^{-1}\ g(s_1 + a_1, \ldots, s_r + a_r)$$

$$= L\left\{ exp\left(- \sum_{i=1}^{r} a_i x_i \right) f(x_1, \ldots, x_r)\ ;\ s_1, \ldots, s_r \right\} \qquad \ldots (2.3)$$

Also, by virtue of (1.1) and ([6], *p.* 251, *Eq.* (*C*.1)), on interchanging the $(\xi_1, \ldots, \xi_r)$ – and $(x_1, \ldots, x_r)$ – integrals and evaluating the $(x_1, \ldots, x_r)$ – integral with the help of a known result ([8], p. 126, Eq (3.8)), we get

$$(s_1^{-\rho_1+1} \ldots s_r^{-\rho_r+1}\ H_1\left[y_1 s_1^{-\omega_1}, \ldots, y_r s_r^{-\omega_r} \right]$$

$$= L\Big\{ x_1^{\rho_1 - 1} \ldots x_r^{\rho_r - 1}$$

$$.\,H^{O,\,O\,:\,m_1,\,n_1\,;\,\ldots\,;\,m_r,\,n_r}_{p,\,q\,:\,p_1,\,q_1+1\,;\ldots;\,p_r,\,q_r+1}\left[\begin{matrix} y_1 x_1^{-\omega_1} \\ \vdots \\ y_r x_r^{\omega_r} \end{matrix} \left| \begin{matrix} (a_j\,;\,\alpha_j', \ldots, \alpha_j^{(r)})_{1,p} : \\ (b_j\,,\,\beta_j', \ldots, \beta_j^{(r)})_{1,q} : \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (c_j', \gamma_j')_{1,p_1}\,;\ldots;\,(c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ (d_j', \delta_j')_{1,q_1}, (1-\rho_1, \omega_1)\,;\ \ldots\,;\,(d_j^{(r)}\,;\,\delta_j^{(r)})_{1,q_r}, (1-\rho_r, \omega_r) \end{matrix} \right]$$

$$;\ s_1, \ldots, s_r \Big\} \qquad \ldots (2.4)$$

where the sets of conditions (ii) and (iii) mentioned with the main theorem are satisfied.

Now applying (1.3) for the transform pairs (2.3) and (2.4) and interpreting the left-hand side of the result thus obtained with the help of (2.1) , we easily arrive at the result (2.2) .

3. SPECIAL CASES

The main theorem established here involves the H-function of several variables, which is one of the most general functions introduced so far and includes almost all named special functions of one or more variables . Thus, with the help of results given by R. S. Garg ([2] , *pp*. 57-61) and Srivastava, Gupta and Goyal ([6] , *pp* . 18-19) , one can easily obtain a number of new and interesting theorems by suitably specializing the parameters of the H-functions of several variables . To illustrate this, we give below two interesting special cases of the main theorem .

If in the main result we take $p = q = d_j^{(i)} = 0$, $n_i = m_i = p_i = q_j = \gamma_1^{(i)} = \delta_1^{(i)} = \omega_i = 1$ and $c_1^{(i)} = 1 - \alpha_i$ ($i = 1, \ldots, r$) , we get the following corollary :

Corollary 1. *For the functions* $f(x_1, \ldots, x_r)$ *and* $g(x_1, \ldots, x_r)$ *related by* (1.2) , *let*

$$h(s_1, \ldots, s_r) = L\left\{ \prod_{i=1}^{r} \left(\Gamma(\alpha_i)\, x_i^{-\rho_i+\alpha_i} (x_i+y_i)^{-\alpha_i} \right) f(x_1, \ldots, x_r) \; ; \; s_1, \ldots, s_r \right\} \qquad \ldots (3.1)$$

Then

$$h(s_1, \ldots, s_r) = (s_1 \ldots s_r) \frac{\Gamma(\alpha_1) \ldots \Gamma(\alpha_r)}{\Gamma(\rho_1) \ldots \Gamma(\rho_r)} \int_0^\infty \ldots \int_0^\infty$$

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} \left\{ x_i^{\rho_i - 1} (x_i + s_i)^{-1} \; {}_1F_1 \, [\alpha_i ; \rho_i ; - y_i x_i] \right\}$$

$$g (x_1 + s_1 , \ldots , x_r + s_r) \, dx_1 \ldots dx_r , \qquad .. (3.2)$$

provided that the Laplace transforms of $| g (x_1 , \ldots , x_r) |$ *and*

$\left| \prod_{i=1}^{r} \left\{ x_i^{-\rho_i + \alpha_i} (x_i + y_i)^{-\alpha_i} \right\} g (x_1 , \ldots , x_r) \right|$ *exist , and the multiple integral on the right-hand side of (3.2) is absolutely convergent .*

Next, putting $p = q = 0$ and $r = 1$ in the main theorem, we get the one-variable case given by

Corollary 2. *If*

$$g (s) = L \{ f (x) ; s \} \qquad \ldots (3.3)$$

and

$$h (s) = L \left\{ x^{-\rho} f (x) \; H_{p,q}^{m,n} \left[y x^{-\omega} \,\middle|\, \begin{matrix} (c_j , \gamma_j)_{1,p} \\ (d_j , \delta_j)_{1,q} \end{matrix} \right] ; s \right] , \qquad .. (3.4)$$

then

$$h (s) = s \int_0^\infty x^{\rho - 1} (x + s)^{-1} g (x + s) \; H_{p,q+1}^{m,n} \left[y x^{\omega} \,\middle|\, \begin{matrix} (c_j , \gamma_j)_{1,p} \\ (d_j , \delta_j)_{1,q} , (1 - \rho , \omega) \end{matrix} \right] dx \qquad \ldots (3.5)$$

valid under conditions derivable from those for the main theorem .

The theorems established earlier by Saxena ([5], *p*. 230 *Eq.* (2)), Maloo ([4], *p.* 844, *Eq.* (8)) and several others follow as its particular cases, if we reduce the H-function involved in corollary 2 to the appropriate special functions occurring in these theorems .

4. APPLICATIONS

Now we shall obtain a very general and interesting integral with the help of the main theorem . Taking,

(4.1) $f(x_1,\ldots,x_r) =$

$$= \prod_{i=1}^{r} \left\{ x_i^{\lambda_i-1} H_{P_i,Q_i+1}^{M_i,0} \left[b_i x_i^{t_i} \left| \begin{matrix} (e_j^{(i)},E_j^{(i)})_{1,P_i} \\ (f_j^{(i)},F_j^{(i)})_{1,Q_i}, (1-\lambda_i,t_i) \end{matrix} \right. \right] \right.$$

in (1.2) , we get with the help of a known result ([6] , *p.* 16, *Eq.* (2.4.2)) :

(4.2) $g(s_1,\ldots,s_r) =$

$$= \prod_{i=1}^{r} \left\{ s_i^{-\lambda_i+1} H_{P_i,Q_i}^{M_i,0} \left[b_i s_i^{-t_i} \left| \begin{matrix} (e_j^{(i)},E_j^{(i)})_{1,P_i} \\ (f_j^{(i)},F_j^{(i)})_{1,Q_i} \end{matrix} \right. \right] \right.$$

where

$$U^{(i)} = \sum_{j=1}^{M_i} F_j^{(i)} - \sum_{j=M_i+1}^{Q_i} F_j^{(i)} - \sum_{j=1}^{P_i} E_j^{(i)} - t_i > 0,$$

$$|\arg(b_i)| < \tfrac{1}{2} U^{(i)}\pi, \text{ and } \min_{1\leqslant i\leqslant r} \{ Re(s_i), Re(\lambda_i), t_i \} > 0.$$

Making use of (2.1) , (4.1) and (1.1) , we easily get

(4.3) $$h(s_1,\ldots,s_r) = \int_0^{\infty}\cdots\int_0^{\infty} \prod_{i=1}^{r} \left\{ s_i x_i^{\lambda_i-\rho_i-1} e^{-s_i x_i} H_{P_i,Q_i+1}^{M_i,0} \right.$$

$$\left[b_i x_i^{t_i} \left| \begin{matrix} (e_j^{(i)},E_j^{(i)})_{1,P_i} \\ (f_j^{(i)},F_j^{(i)})_{1,Q_i}, (1-\lambda_i,t_i) \end{matrix} \right. \right] H_1 \left[y_1 x_1^{-\omega_1},\ldots, y_r x_r^{-\omega_r} \right]$$

$dx_1\ldots dx_r$

To evaluate the multiple integral on the right-hand side of (4.3), write out the exponential series. interchange the order of integration and summation (which is justified under the conditions stated below), evaluate the ($x_1, \ldots, x_r$) — integral, so obtained with the help of a known result ([2], p. 83) . We thus obtain

(4·4) $h(s_1, \ldots, s_r)=$

$$=\left(\frac{s_1 \ldots s_r}{t_1 \ldots t_r}\right) \prod_{i=1}^{r} \left\{ b_i^{(\rho_i-\lambda_i)/t_i} \sum_{k_i=0}^{\infty} \frac{(-s_i b_i^{-1/t_i})^{k_i}}{k_i\,!} \right\}$$

$$. H^{0,0:\, m_1+M_1,\, n_1;\, \ldots;\, m_r+M_r,\, n_r}_{p,q:\, p_1+P_1,\, q_1+Q_1+1;\, \ldots;\, p_r+P_r,\, q_r+Q_r+1} \left[\begin{matrix} y_1 b_1^{w_1/t_1} \\ \vdots \\ y_r b_r^{w_r/t_r} \end{matrix}\,\middle|\, \begin{matrix} U \\ V \end{matrix}\right],$$

where U stands for

$$(a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,p} : (c_j',\gamma_j')_{1,p_1},\ (e_j'+\frac{\lambda_1-\rho_1+k_1}{t_1}E_j',$$

$$E_j'\frac{w_1}{t_1})_{1,P_1};\ \ldots;\ (c_j^{(r)},\gamma_j^{(r)})_{1,P_r},$$

$$(e_j^{(r)}+\frac{\lambda_r-\rho_r+k_r}{t_r}E_j^{(r)},\ \frac{w_r}{t_r}E_j^{(r)})_{1,P_r}$$

and V stands for

$$(b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,q}\ :\ (d_j',\delta_j')_{1,m_1},$$

$$(f_j'+F_j'\frac{\lambda_1-\rho_1+k_1}{t_1},\ F_j'\frac{w_1}{i_1})_{1,Q_1},\ (d_j',\delta_j')_{m_1+1,q_1},$$

$$(1-\rho_1+k_1,\ w_1);\ \ldots;\ (d_j^{(r)},\delta_j^{(r)})_{1,m_r},$$

$$(f_j^{(r)} + F_j^{(r)} \frac{\lambda_r - \rho_r + k_r}{t_r} , F_j^{(r)} \frac{w_r}{t_r})_{1,Q_r} , (d_j^{(r)} , \delta_j^{(r)})_{m_r+1, q_r} ,$$

$$(1 - \rho_r + k_r , w_r)$$

provided that $\{ Re \ (s_i) , w_i , t_i \} > 0$,

$U^i > 0$, $| \arg (b_i) | < \frac{1}{2} U^{(i)} \pi (i = 1 , \ldots , r)$

$$Re (\lambda_i - \rho_i) + t_i \min_{1 \leqslant j \leqslant m_i} Re (f_j^{(i)} / F_j^{(i)}) - \omega_i$$

$$\max_{1 \leqslant j \leqslant n_i} Re \{ (c_j^{(i)} - 1) / \gamma_j^{(i)} \} > 0$$

and the multiple series on the right-hand side is absolutely convergent.

Substituting these values of $g (x_1 , \ldots , x_r)$ and $h (s_1 , \ldots , s_r)$ from (4.2) and (4.4), respectively , in (2.2), altering the parameters of the H-function of several variables occuring on the right-hand side of (2.2) slightly, we arrive at the following interesting integral after a litile simplification .

$$(4.5) \quad \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} \left\{ x_i^{\rho_i - 1} (x_i + s_i)^{-\lambda_i} H_{P,Q_i}^{M_i,0} \left[b_i (x_i + s_i)^{-t_i} \middle| \begin{matrix} (e_j^{(i)} , E_j^{(i)})_{1,P_i} \\ (f_j^{(i)} , F_j^{(i)})_{1,Q_i} \end{matrix} \right] \right. . H_1 \left[y_1 x_1^{\omega_1} , \ldots , y_r x_r^{\omega_r} \right] dx_1 \ldots dx_r$$

$$= \prod_{i=1}^{r} \left\{ \frac{b_i^{(\rho_i - \lambda_i)/t_i}}{t_i} \right\} \sum_{k_i=0}^{\infty} \frac{(-1)^{k_i} (s_i b_i^{-1/t_i})^{k_i}}{k_i !}$$

$$. H_{p,q : p_1+P_1+1, q_1+Q_1+1 ; \ldots ; p_r+P_r+1 , q_r+Q_r+1}^{0,0 ; m_1+M_1, n_1+1 ; \ldots ; m_r+M_r , n_r+1} \left[\begin{matrix} y_1 b_1^{w_1/t_1} \\ \vdots \\ y_r b_r^{w_r/t_r} \end{matrix} \middle| \begin{matrix} U' \\ \\ V' \end{matrix} \right]$$

where U' stands for

$$(a_j\,;\alpha_j'\,;\ \ldots,\alpha_j^{(r)})_{1,p}\ :(1-\rho_1,w_1),\ (c_j',\gamma_j')_{1,p_1},$$

$$\left(e_j'+\frac{\lambda_1-\rho_1+k_1}{t_1}E_j',\ \frac{w_1}{t_1}E_j'\right)_{1,P_1};\ldots;(1-\rho_r,w_r),\ (c_j^{(r)},\gamma_j^{(r)})_{1,p_r},$$

$$\left(e_j^{(r)}+\frac{\lambda_r-\rho_r+k_r}{t_r}E_j^{(r)},\ \frac{w_r}{t_r}E_j^{(r)}\right)_{1,P_r}$$

V' stands for

$$(b_j\,;\beta_j',\ldots,\beta_j^{(r)})_{1,q_1}\ :(d_j',\delta_j')_{1,m_1},$$

$$\left(f_j'+\frac{\lambda_1-\rho_1+k_1}{t_1}F_j'\ \frac{w_1}{t_1}F_j'\right)_{1,Q_1},$$

$$(d_j',\delta_j')_{m_1+1,\,q_1},\ (1-\rho_1+k_1,w_1);\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,m_r},$$

$$\left(f_j^{(r)}+\frac{\lambda_r-\rho_r+k_r}{t_r}F_j^{(r)},\ \frac{w_r}{t_r}F_j^{(r)}\right)_{1,Q_r},\ (d_j^{(r)},\delta_j^{(r)})_{m_r+1,\,q_r},$$

$$(1-\rho_r+k_r,w_r)\ .$$

$Re(\rho_i)>0$; and the set of conditions mentined with (4.4) are satisfied .

The integral (4.5) is believed to be one of the most general integrals evaluated so far ; it isc apable of yielding a number of interesting integrals involving simpler functions of one , two or more variables as its particular cases .

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