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ON PLANE WAVES OF SOUND IN A POROUS MEDIUM

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ABSTRACT

The effects of viscosity and thermal conductivity on plane wave propagation through compressible fluid in a rigid, homogeneous and isotropic porous medium are considered. The amplitude of the waves attenuates exponentially as they propagate. This attenuation is increased with increases in viscosity, thermal conductivity and frequency and decreases in the permeability.

1. INTRODUCTION

A detailed account of the propagation of plane waves of sound in the presence of viscosity and heat conduction effects has been given by Lamb [3] and Rayleigh [5]. The action of porous bodies in the absorption of sound for the important though special case of waves in tubes has also been treated by Lamb [3] and Rayleigh [5]. Morse [4] has extended this study and compared his theoretical results of the attenuation with the experimental values. However, he made several assumptions on the particle sizes and frequency ranges.

The problem needs to be studied in its full generality. A re-examination of the effects of viscosity and thermal conductivity on the wave propagation through a compressible fluid in a rigid, homogeneous and isotropic porous medium is certainly called for and is the object of the present paper.

2. PERTURBATION EQUATIONS

Here we consider the plane wave propagation in x-direction through a porous medium. The porous medium is assumed to be rigid, homogeneous and isotropic. Then the equations of motion and continuity governing the flow of viscous, compressible fluid through a porous medium are (Joseph [2], Yih [6]) :

$$\frac{1}{\epsilon}\left(\frac{\partial u}{\partial t}+\frac{1}{\epsilon}u\frac{\partial u}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{4\mu}{3\rho\epsilon}\frac{\partial^2 u}{\partial x^2}-\frac{\mu}{\rho k_1}u\,, \quad (1)$$

and

$$\epsilon\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0 \quad (2)$$

Here u, p, ρ and μ denote respectively the velocity, pressure, density and viscosity of the fluid wereas ϵ and k_1 stand for medium porosity and medium permeability. For a unit volume of the porous medium, the amount of heat required to produce small changes in the volume V and temperature T (Lamb [3], Joseph [2]) is

$$\delta Q=\frac{\rho_0 C_v(\gamma-1)T_0}{V_0}\,\delta V+\left[\rho_0 C_v\epsilon+\rho_s C_s(1-\epsilon)\right]\delta T\,, \quad (3)$$

where ρ_0, C_v ; ρ_s, C_s stand for the density and the specific heat at constant volume of the fluid and the solid respectively.

$\gamma = \left(\frac{C_p}{C_v}\right)$ denotes the ratio of specific heats of the fluid.

Multiplying (3) by δx and differentiating with respect to t. we obtain the rate at which heat must be supplied to the stratum and this on equating to $K \frac{\partial^2 T}{\partial x^2} \delta x$ gives

$$E \frac{\partial T}{\partial t} + (\gamma - 1) \frac{T_0}{V_0} \frac{\partial V}{\partial t} = x \frac{\partial^2 T}{\partial x^2} , \tag{4}$$

where K is the 'effective' thermal conductivity,

$x = \left(\frac{K}{\rho_0 C_v}\right)$ is the thermal diffusivity and $E = \epsilon + \frac{(1-\epsilon)\rho_s C_s}{\rho_0 C_v}$.

The macroscopic heat conduction eq. (4) can also be obtained by an averaging procedure starting from the microscopic scale (inside the pore space) to the macroscopic one (Bear [1]).

The equation of state is

$$\frac{p}{p_0} = \frac{\rho T}{\rho_0 T_0} \tag{5}$$

Substituting

$$\rho = \rho_0 (1+S), \quad T = T_0 (1+\eta), \tag{6a, b}$$

and retaining only the linear terms, eqs. (1), (2), (4) and (5) give

$$\frac{1}{\epsilon} \frac{\partial u}{\partial t} = - \frac{1}{\rho_0} \frac{\partial p}{\partial x} + \frac{4\nu}{3\epsilon} \frac{\partial^2 u}{\partial x^2} - \frac{\nu}{k_1} u , \tag{7}$$

$$\epsilon \frac{\partial s}{\partial t} = - \frac{\partial u}{\partial x} , \tag{8}$$

$$E \frac{\partial \eta}{\partial t} - (\gamma - 1) \frac{\partial s}{\partial t} = x \frac{\partial^2 \eta}{\partial x^2} , \tag{9}$$

$$p = p_0 (1+s+\eta) , \tag{10}$$

where ν ($= \mu/\rho$) is the kinematic viscosity of the fluid .

Equations (7) to (10) yield

$$\frac{1}{\epsilon} \frac{\partial^2 u}{\partial t^2} = \frac{b^2}{\epsilon} \frac{\partial^2 u}{\partial x^2} - b^2 \frac{\partial^2 \eta}{\partial x \partial t} + \frac{4\nu}{3\epsilon} \frac{\partial^3 u}{\partial x^2 \partial t} - \frac{\nu}{k_1} \frac{\partial u}{\partial t} , \tag{11}$$

and $$E \frac{\partial \eta}{\partial t} + \frac{\gamma - 1}{\epsilon} \frac{\partial u}{\partial x} - x \frac{\partial^2 \eta}{\partial x^2} , \tag{12}$$

where b^2 ($= p_0/\rho_0$) denotes the square of the Newtonian velocity of sound .

3. DISCUSSION

Assuming that u and η both vary as

$$a \; exp \; (i\sigma t + kx) , \tag{13}$$

where k, σ and a denote the wave number , growth rate and amplitude respectively and eliminating η between (11) and (12), we obtain the dispersion relation

$$x \left(ib^2 - \frac{4\nu\sigma}{3} \right) k^4 + \left[E\sigma \left(b^2 + \frac{4i\nu\sigma}{3} \right) + x \left(i\sigma^2 + \frac{\nu\epsilon\sigma}{k_1} \right) + b^2\sigma \left(\gamma - 1 \right) \right] k^2 + E \left(\sigma^3 - \frac{i\nu\epsilon\sigma^2}{k_1} \right) = 0 . \tag{14}$$

Now several cases of interest are derived and discussed .

(i) For fluid continuum ($\epsilon = 1$, $E = 1$, $k_1 \to \infty$) and in the absence of dissipative effects of viscosity and thermal conductivity

($\nu = x = 0$), eq . (14) gives

$$k = \pm \frac{i\sigma}{c} , \tag{15}$$

where c ($\sqrt{\gamma p_0/\rho_0}$) denotes the velocity of sound .

(ii) For fluid continuum, $\nu = 0$ (nonviscous fluid) and (practically isothermal conditions). eq . (14) yields

$$k = \pm \frac{i\sigma}{b} \tag{16}$$

(iii) For fluid continuum where only dissipation due to viscosity is considered (*i. e.* $x = 0$) , eq . (14) yields

$$k = -\frac{i\sigma}{c} - \frac{2}{3} \frac{\nu\sigma^2}{c^3} , \tag{17}$$

where negative sign has been taken and $\nu\sigma / c^2$ being small, its square and higher powers have been neglected .

Substituting this value of k in (13), we obtain, for the waves propagated in the direction of x-positive .

$$u = Re \left\{ a\, e^{-x/l} \, . \, e^{i\sigma (t - \frac{x}{c})} \right\} = a\, e^{-x/l} \cos \sigma \left(t - \frac{x}{c} \right) , \tag{18}$$

where Re denotes the real part and $l = \frac{3}{2} \frac{c^3}{\nu\sigma^2}$.

Thus the amplitude of the waves decreases exponentially as x increases and if σ is large the diminution is more rapid . To the first order of $\nu\sigma/c^2$, the wave velocity is unaffected by the dissipation.

(iv) For porous medium where dissipation due to viscosity alone is considered (*i. e.* $x = 0$), eq . (14) gives

$$k = \pm \frac{i\sigma}{b}\left(\frac{E}{\gamma-1+E}\right)^{1/2}\left(1-\frac{i\nu\epsilon}{k_1\sigma}\right)^{1/2}\left(1-\frac{2i\nu\sigma E/3b^2}{\gamma-1+E}\right), \quad (19)$$

square and higher powers of $\nu\sigma / c^2$ being neglected . For large $k_1\sigma$ and taking the lower sign, eq . (19) gives

$$k = -\frac{i\sigma}{b\left(\frac{\gamma-1+E}{E}\right)^{1/2}}\left[1-\frac{\nu^2\epsilon}{3k_1 b^2\left(\frac{\gamma-1+E}{E}\right)}\right]$$

$$-\frac{\nu}{b\left(\frac{\gamma-1+E}{E}\right)^{1/2}}\left[\frac{\epsilon}{2k_1}+\frac{2\sigma^2}{3b^2\left(\frac{\gamma-1+E}{E}\right)}\right]. \quad (20)$$

Substitute this value of k in (13) and take the real part, we obtain, for the waves propagated in the direction of x-positive,

$$u = a e^{-x/l} \cdot \cos\sigma\left[t-\frac{x}{b\left(\frac{\gamma-1+E}{E}\right)^{1/2}}\left\{1-\frac{\nu^2\epsilon}{3k_1 b^2\left(\frac{\gamma-1+E}{E}\right)}\right\}\right], \quad (21)$$

where

$$l = \frac{b\left(\frac{\gamma-1+E}{E}\right)^{1/2}}{\nu\left[\frac{\epsilon}{2k_1}+\frac{2\sigma^2}{3b^2\left(\frac{\gamma-1+E}{E}\right)}\right]}. \quad (22)$$

Equations (21) and (22) imply that the amplitude of the waves diminish exponentially as they proceed , the diminution being more rapid the greater the values of frequency and fluid viscosity but smaller the value of permeability.

(v) Finally, we consider the case of porous medium where only the effect of thermal conductivity is considered (*i. e.* $\nu = 0$), eq. (14) yields

$$k^2 = -\frac{1}{2ix}\left\{\sigma(\gamma-1+E) + \frac{ix\sigma^2}{b^2}\right\} \pm \frac{1}{2ix}$$

$$\left[\left\{\sigma(\gamma-1+E) + \frac{ix\sigma^2}{b^2}\right\}^2 - \frac{4ixE}{b^2}\sigma^3\right]^{1/2} \tag{23}$$

Since the ratio $x\sigma/c^2$ is small for ordinary sound waves, the roots of (23) are approximately

$$k_a^2 = -\frac{\sigma^2}{b^2\frac{(\gamma-1+E)}{E}}, \tag{24}$$

and

$$k_b^2 = \frac{i\sigma}{x}(\gamma-1+E). \tag{25}$$

A more accurate value of k_a^2 is

$$k_a^2 = -\frac{\sigma^2}{b^2\frac{(\gamma-1+E)}{E}}\left[1 - \frac{ix\sigma}{b^2(\gamma-1+E)}\left\{1 - \frac{1}{\frac{(\gamma-1+E)}{E}}\right\}\right]. \tag{26}$$

Therefore

$$k_a = \pm\left(\frac{i\sigma}{b\sqrt{\frac{(\gamma-1+E)}{E}}} + \frac{1}{l}\right). \tag{27}$$

where

$$
l = \frac{b^3\,(\gamma-1+E)\left(\dfrac{\gamma-1+E}{E}\right)^{1/2}}{\dfrac{1}{2}\,x\sigma^2\left\{1-\dfrac{1}{\dfrac{(\gamma-1+E)}{E}}\right\}} \,. \tag{28}
$$

The complete solution for $x > 0$ is then

$$
u = a\,e^{i\sigma t + k_a x} + d e^{i\sigma t + k_b x}, \tag{29}
$$

provided k_a, k_b are chosen so that their real parts are negative, a and d being the amplitudes of the respective waves. The second term in the value of u is insignificant in comparison with the first term. Substituting (27) in (29), it is seen that the amplitude of the waves decreases exponentially as they propagate and this attenuation is more rapid the greater the values of frequency and thermal conductivity.

A similar analysis holds for the simultaneous presence of the viscosity and thermal conductivity effects. The effects (decrease in amplitude of the waves) are increased but remain of the same order of magnitude.

Hence we conclude the whole analysis in the following statements. The waves attenuate exponentially as they propagate and the attenuation increases with frequency and the inverse of permeability. The rate of attenuation of the waves also increases with increases in the values of viscosity and thermal conductivity.

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