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THERMAL CONVECTION IN A POROUS MEDIUM IN THE PRESENCE OF SUSPENDED PARTICLES

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ABSTRACT

The effect of suspended particles is considered on thermal convection in a porous medium. Principle of exchange of stabilities is satisfied. The effect of suspended particles is to destabilize the layer. The medium permeability also has a destabilizing effect on the system. The effect of uniform rotation is also considered and is found to stabilize a certain wave-number range in thermal convectian in porous medium, which were unstable in the absence of rotation.

1. INTRODUCTION

A comprehensive account of the thermal convection problem, under varying assumptions of hydrodynamics, has been given by Chandrasekhar [1]. Lapwood [2] has studied the stability of convective flow in a porous medium using Rayleigh's procedure. The Rayleigh instability of a thermal boundary layer in flow through a porous medium has been considered by Wooding [5] . The gross effect, when the fluid slowly percolates through the pores of the rock, is represented by the Darcy's law .

The fluid may not be absolutely pure but may, instead, be permeated with suspended (or dust) particles , The effect of particle mass and heat capacity on the onset Bénard convection has been considered by Scanlon and Segel [3]. The effect of suspended particles was found to destabilize the layer . In another context. Palaniswamy and Purushotham [4] have studied the stability of shear flow of stratified fluids with fine dust and have found the effect of fine dust to increase the region of instability .

Motivated by interest in fluid-particle mixtures and keeping in mind the geophysical situations occurring below the Earth's surface, we set out to study the effect of suspend particles on thermal convection in a porous medium . The effect of uniform rotation on the problem has also been considered due to the importance of the Coriolis force .

2. DESCRIPTION OF THE PROBLEM AND PERTURBATION EQUATIONS

Here we consider a horizontal fluid - particle layer of depth d unbounded in the horizontal (x, y) directions and bounded by the planes $z=0$ and $z=d$ in a porous medium . This layer is heated from below such that a steady adverse temperature gradient β $(= | dT/dz |)$ is maintained . The equations of motion and continuity for the fluid are

$$\frac{\rho}{\epsilon}\left[\frac{\partial \mathbf{u}}{\partial t}+\frac{1}{\epsilon}\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\right]=-\nabla p-g\rho\vec{\lambda}-\frac{\mu}{k_1}\mathbf{u}+\frac{kN}{\epsilon}\left(\mathbf{v}-\mathbf{u}\right), \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 . \qquad (2)$$

Here ρ , μ $(=\rho\nu)$, p and $\mathbf{u}$ (u, v, w) denote respectively the density, the viscosity, the pressure and the velocity of the pure fluid. $\mathbf{v}$ (l, r, s) and N $(\bar{x}, t)$ denote the velocity and number density of the particles. ϵ is the medium porosity, k_1 is the medium permeability, g is the acceleration due to gravity, $\bar{x} = (x, y, z)$, $\vec{\lambda} = (0, 0, 1)$ and $K = 6\pi\mu\eta$, η being the particle radius , is the Stokes' drag . Assuming uniform particle size, spherical shape and small relative velocities between the fluid and particles, the presence of particles adds an extra force term in the equations of motion (1), proportional to the velocity difference between particles and fluid .

The force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid . The buoyancy force on the particles is negligibly small . Interparticle reactions are ignored for we assume that the distances between particles are quite large compared with their diameter . The equations of motion and continuity for the particles are

$$mN\left[\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\epsilon}\left(\mathbf{v} \cdot \nabla\right)\mathbf{v}\right] = kN\left(\mathbf{u} - \mathbf{v}\right) - \epsilon mNg\vec{\lambda}, \qquad (3)$$

$$\epsilon \frac{\partial N}{\partial t} + \nabla \cdot \left(N\mathbf{v}\right) = 0 , \qquad (4)$$

where mN is the mass of particles per unit volume .

Let C, Cp, T and q denote respectively the heat capacity of fluid at constant pressure, the heat capacity of particles, the temperature and the "effective" thermal conductivity is the conductivity of the clean fluid . If the particles and the fluid are assumed to be in thermal equilibrium, then the heat conduction equation gives

$$\left[\rho C\epsilon + \rho_s C_s(1-\epsilon)\right]\frac{\partial T}{\partial t} + \rho C(\mathbf{u}\cdot\nabla)T + mNC_p\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)T$$

$$= q\nabla^2 T, \qquad (5)$$

where ρ_s, C_s are the density and the heat capacity of the solid matrix respectively . The equation of state for the fluid is

$$\rho = \rho_0[1-a(T-T_0)], \qquad (6)$$

where ρ_0 is the mean density of the clean fluid and a is the coefficient of thermal expansion .

The initial state of the system, denoted by subscript 0, is taken to be a quiescent layer with a uniform particle distribution N_0 . The initial state

$$\mathbf{u}_0 = 0, \ \mathbf{v}_0 = 0, \ T_0 = -\beta z, \qquad N_0 = \text{constant},$$

is an exact solution to the governing equations .

Let $\delta\rho$, δp, N, θ, $\mathbf{u}$ and $\mathbf{v}$ denote, respectively, the perturbations in density ρ_0, pressure p_0, number density N_0, temperature T, fluid velocity (zero initially) and particle velocity (zero initially) . We scale the physical variables using

d, $\frac{d^2}{k}$, $\frac{k}{d}$, $\frac{\rho\nu k}{d^2}$ and βd as the length, time, velocity, pressure and temperature scale factors respectively . Then the linearized dimensionless perturbation equations are

$$N_p^{-1} \frac{\partial \mathbf{u}}{\partial t} = -\nabla \delta p + N_R \theta \vec{\lambda} - P^{-1} \mathbf{u} + \omega (\mathbf{v} - \mathbf{u}), \tag{7}$$

$$\tau \frac{\partial \mathbf{v}}{\partial t} = \mathbf{u} - \mathbf{v}, \tag{8}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \frac{\partial M}{\partial t} + \nabla \cdot \mathbf{v} = 0, \tag{9}$$

$$E(1 + h) \frac{\partial \theta}{\partial t} = (W + hs) + \nabla^2 \theta \tag{10}$$

In writing eq. (7), use has been made of the Boussinesq equation of state $\delta\rho = -\rho_0 a \theta$. w and s are the vertical fluid and particle velocities, ν is the kinematic viscosity of the fluid, k is the thermal diffusivity, $N_p = \epsilon\nu/k$ is the modified Prandtl number, $N_R = ga\beta d^4/\nu k$ is the Rayleigh number, $f = mN_0/\rho_0 = \tau w N_p$ is the mass fraction, $\omega = KN_0 d^2 / \rho_0 \nu \epsilon$, $P = k_1 / d^2$, $\tau = mk / kd^2$, $h = fC_p / C$, $M = N / N_0$ and $E = \epsilon + (1 - \epsilon) \rho_s C_s / \rho C$.

Let us consider the case of two free surfaces having uniform temperature. The case of two free surfaces, though artificial in nature, enables us to find the analytical solution. The boundary conditions appropriate for the problem are

$$W = \frac{\partial^2 w}{\partial z^2} = \theta = 0 \qquad \text{at } z = 0 \text{ and } 1. \tag{11}$$

Eliminating $\mathbf{v}$ and δp the fluid and heat equations give

$$\left(L_1 + \frac{L_2}{P} \right) \nabla^2 w = L_2 N_R \nabla_1^2 \theta, \tag{12}$$

$$L_2 \left(EH \frac{\partial}{\partial t} - \nabla^2 \right) \theta = \left(\tau \frac{\partial}{\partial t} + H \right) W, \tag{13}$$

where

$$L_1 = N_p^{-1}\left(\tau \frac{\partial^2}{\partial t^2} + F \frac{\partial}{\partial t}\right), \quad L_2 = \left(\tau \frac{\partial}{\partial t} + 1\right),$$

$$F = f + 1, H = h + 1, \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .$$

Eliminating θ between eqs. (12) and (13), we obtain

$$\left(L_1 + \frac{L_2}{P}\right)\left(EH \frac{\partial}{\partial t} - \nabla^2\right) \nabla^2 \omega = N_R \left(\tau \frac{\partial}{\partial t} + H\right)\nabla_1^2 \omega \tag{14}$$

Analyze the perturbations in terms of normal modes by seeking solutions whose dependence on x, y and t is given by

$$w = W(z) \exp(i\alpha_x x + i\alpha_y y + nt), \tag{15}$$

where $\alpha = \left(\alpha_x^2 + \alpha_y^2\right)^{1/2}$ is the wavenumber of the disturbance and n is the growth rate.

Equation (14), using (15), becomes .

$$\left(L_1 + \frac{L_2}{P}\right)\left(D^2 - \alpha^2\right)\left(D^2 - \alpha^2 - EHn\right) W = \left(\tau n + H\right) N_R \alpha^2 W, \tag{16}$$

where

$$L_1 = N_p^{-1} (\tau n^2 + Fn), \quad L_2 = (1 + \tau n) \quad \text{and } D = \frac{d}{dz} .$$

3. FROOF OF THE PRINCIPLE OF EXCHANGE OF STABILITIES

In this section we shall show that, for the problem under discussion, the principle of exchange of stabilities is valid, *i. e.* n is real and the marginal states are characterized by $n = 0$.

Let $$U = (D^2 - a^2) W , \tag{17}$$

and $$X = \left(L_1 + \frac{L_2}{P} \right) U . \tag{18}$$

The equation satisfied by w, in terms of X, is

$$(D^2 - \alpha^2 - Hn) X - a^2 (\tau n + H) N_R W . \tag{19}$$

Multiply eq. (19) by X^*, the complex conjugate of X and integrate over the range of z. We have

$$\int_0^1 X^* (D^2 - \alpha^2 - Hn) X dz = \alpha^2 (\tau n + H) N_R \int_0^1 X^* W dz . \tag{20}$$

Integrating by parts and using eqs. (17), (18) and boundary conditions (11), eq. (20) yields

$$I_1 + \left(\alpha^2 + EHn \right) I_2 = \alpha^2 N_R \left(\tau n + H \right) \left(L_1^* + \frac{L_2^*}{P} \right) I_3 , \tag{21}$$

where

$$I_1 = \int_0^1 | DX |^2 dz ,$$

$$I_2 = \int_0^1 | x |^2 \, dz , \tag{22}$$

$$I_3 = \int_0^1 (| DW |^2 + \alpha^2 | W |^2) \, dz ,$$

which are all positive definite . Putting $n = in_0$, where n_0 is real, into eq . (21) and equating the imaginary parts, we obtain

$$n_0^2 = - \frac{EHI_2 + \alpha^2 N_R (N_p^{-1} FH + \tau h P^{-1}) I_3}{\alpha^2 N_R N_p^{-1} \tau^2 I_3} , \tag{23}$$

$$\text{or} \qquad n_0 = 0 . \tag{24}$$

Since n_0 is real and integrals $I_1 - I_3$ are positive definite, it follows that $n_0 = 0$. Therefore n is real and the principle of exchange of stabilities is valid for the problem under consideration .

4. DISPERSION RELATION AND DISCUSSION

It has been proved above that marginal state is that of stationary convection and the principle of exchange of stabilities is valid . When instability sets in as stationary convection , the marginal state will be characterized by $n = 0$ and eq. (16) reduces to

$$\frac{1}{P} (D^2 - \alpha^2)^2 \, W = H \alpha^2 N_R W . \tag{25}$$

Here we consider the case of two free boundaries . It can be shown that all the even order derivatives of W vanish on the boundaries and hence the proper solution of (25) characterizing the lowest mode is

$$W = W_0 \sin \pi z , \tag{26}$$

where W_0 is a constant .

Substituting the solution (26) in eq. (25), we obtain

$$N_R = \frac{(\pi^2 + \alpha^2)^2}{\alpha^2 HP} \tag{27}$$

Equation (27) yields

$$\frac{dN_R}{dp} = - \frac{(\pi^2 + \alpha^2)^2}{\alpha^2 HP^2}, \tag{28}$$

and

$$\frac{dN_R}{dH} = - \frac{(\pi^2 + \alpha^2)^2}{\alpha^2 PH^2}. \tag{29}$$

The medium permeability has a destabilizing effect on the system, The effect of suspended particles is also to destabilize the layer .

5. EFFECT OF UNIFORM ROTATION

In this section we consider the same problem as described above except that the system is in a state of uniform rotation $\vec{\Omega}\,(0, 0, \Omega)$. The Coriolis force on the particles is neglected . Equations (8) - (10) remain unaltered , Denoting the Taylor number $T_A = 4\Omega^2 d^4 / \nu^2$, the linearized nondimensional perturbation equations of motion for the fluid are

$$N_p^{-1} \frac{\partial u}{\partial x} = - \frac{\partial}{\partial x} \delta p + \omega(l - u) + T_A^{1/2} v - \frac{1}{p} u, \tag{30}$$

$$N_p^{-1} \frac{\partial v}{\partial t} = - \frac{\partial}{\partial y} \delta p + \omega(\gamma - v) - T_A^{1/2} u - \frac{1}{P} v, \tag{31}$$

$$\overset{-1}{N_p}\ \frac{\partial w}{\partial t} = -\ \frac{\partial}{\partial z}\ \delta p + \omega\,(\,\delta - w\,) + N_R\,\theta - \frac{1}{P}\ w\ . \tag{32}$$

Eliminating v and δp between eqs . (30) - (32) and (8) , wc obtain

$$\left(L_1 + \frac{L_2}{P}\right)^2 \nabla^2 w + L_2^2\ T_A \frac{\partial^2 w}{\partial z^2} = L_2\left(L_1 + \frac{L_2}{P}\right) N_R\ \nabla_1^2\ \theta\ . \tag{33}$$

Eliminating θ between (10) and (33) and using expressing (15), we obtain

$$\left(D^2 - \alpha^2 - EH\epsilon\right)\left[\left(L_1 + \frac{L_2}{P}\right)^2 \left(D^2 - \alpha^2\right) + L_2^2\ T_A D^2\right] W$$

$$= \left(L_1 + \frac{L_2}{P}\right) \alpha^2 N_R \left(\tau\epsilon + H\right) W\ . \tag{34}$$

For the stationary convection, $n = 0$ and eq. (34) reduces to

$$\left(D^2 - \alpha^2\right)\left[\frac{1}{P^2}\left(D^2 - \alpha^2\right) + T_A D^2\right] W = \frac{1}{P}\ \alpha^2 H N_R\ W\ . \tag{35}$$

Again considering the case of two free boundaries with constant temperature and using the proper solution (26), eq . (35) yields

$$N_R = \frac{(\,\pi^2 + \alpha^2\,)}{\alpha^2 H}\left[\frac{1}{P}\left(\pi^2 + \alpha^2\right) + \pi^2 T_A P\right]. \tag{36}$$

Equation (36) yields

$$\frac{dN_R}{dP} = \frac{(\,\pi^2 + \alpha^2\,)}{\alpha^2 H}\left[-\ \frac{(\,\pi^2 + \alpha^2\,)}{P^2} + T_A \pi^2\right]\ . \tag{37}$$

If $T_A > \left(1 + \frac{\alpha^2}{\pi^2}\right) / P^2$, $\frac{dN_R}{dp}$ is positive and if

$T_A < \left(1 + \frac{\alpha^2}{\pi^2}\right) / P^2$, $\frac{dN_R}{dP}$ is negative . So the medium permeability has both stabilizing and destabilizing effects depending on the Taylor number . The rotation, thus, stabilizes a certain wave-number range in thermal convection in a porous medium in the presence of suspendde particles, which were unstable in the absence of rotation .

Equation (36) also yields

$$\frac{dN_R}{dT_A} = \frac{P\pi^2 (\pi^2 + \alpha^2)}{\alpha^2 (1 + h)} , \tag{38}$$

and

$$\frac{dN_R}{dh} = - \frac{(\pi^2 + \alpha^2) \left[\frac{1}{P} (\pi^2 + \alpha^2) + \pi^2 T_A P \right]}{\alpha^2 (1 + h)^2} . \tag{39}$$

The rotation has stabilizing effect whereas the suspended particles have destabilizing effect on the system under consideration .

6. DISCUSSION

Owing to great importance and applications in Geophysics, the problem of thermal convection in a porous medium in the presence of suspended particles has been considered in the present paper . As in thermal convection problem, the principle of exchange of stabilities is satisfied for the thermal convection in porous medium in the presence of suspended particles and the oscillatory modes are not allowed . In contrast to the effects of rotation and magnetic field on the thermal convection which introduce oscillatory modes, the presence of suspended particles and medium porosity are unable to bring in overstability . Since the heat capactiy of the pure fluid is supplemented by that of the particles, the effect of suspended particles is to destabilize the layer . The medium permeability also has a destabilizing effect on the system . The rotation stabilizes a certain wave–number range in thermal convection in a porous medium in the presence of suspended particles, which were unstable in the absence of rotation and thus rotation has stabilizing effect on the system .

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