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SOME THEOREMS ON BILATERAL GENERATING FUNCTIONS

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ABSTRACT

In an earlier paper [1], we defined a general polynomial system $S_n^A[(x_m), y]$ with the help of a generating relation involving the Lauricella function F_A. In the present paper, we establish an effective method of obtaining bilateral generating relations in terms of Pochhammer's double-loop integral .

1. INTRODUCTION

The general polynomial system $\{ S_n^A[(x_m), y)/n=0, 1, 2 .. \}$ is defined by means of generating reletion [1] :

$$\sum_{n=0}^{\infty} S_{n,\nu,(\gamma_m;1),(c_m)}^{A;m,a,(b_m),(\mu_m)}[(x_m), y]\, t^n$$

$$= e^{\nu y t} F_A[a, (b_m); (x_m); \mu_1 x_1 t, \mu_2 x_2 t^{r_2}, \ldots, \mu_m x_m t^{r_m}], \quad (1.1)$$

where F_A is one of the Lauricella functions [3, p.41] . The conditions for validity of (1.1) are

(i) (x_m) , y , t are all finite quantities .

(ii) m, $(r_{m;1})$ are natural numbers ;

(iii) ν, (μ_m) are finite numbers ;

(iv) a, (b_m) are non-zero numbers ;

(v) (c_m) are neither zero nor negative integers

For this polynomial system, we have an explicit representation in the form [1] :

$$S_n^A [(x_m) , y] = \sum_{m_1=0}^{n} \sum_{m_2=0}^{[(n-m_1)/r_2]} \cdots \sum_{m_m=0}^{[(n-m_1-\sum_2^{m-1} r_i m_i)/r_m]} \frac{(a)_{n-m_1-\sum_2^m (r_i-1)m_i} (b.)_{n-m_1-\sum_2^m r_i m_i}}{(n-m_1-\sum_2^m r_i m_i)! \ (c_1)_{n-m_1-\sum_2^m r_i m_i}}$$

$$\frac{[(b_{m;1})_{(m_{m;1})}] (\nu y)^{m_1} (\mu_1 x_1)^{n-m_1-\sum_2^m r_i m_i} [(\mu_{m;1} x_{m;1})^{m_{m;1}}]}{[(c_{m;1})_{(m_{m;1})}] [(m_m !)]} \qquad (1.2)$$

Saran [5], Shankar [6], Panda [4], Srivastava ([8], [9]) and Singhal ([7], [12]) and Srivastava and Lavoie [10] have established a number of theorems on bilateral as well as bilinear generating-functions (see also Chapter 8 of the latest work on the subject by Srivastava and Manocha [11]) . Recently, Baweza and Saran [2] gave some interesting theorems for obtaining bilateral generating functions in terms of Pochhammer's double-loop integrals (see, for details, [13,p. 256]) . The object of this paper is to derive analogous generating functions for the polynomial systems defined by (1.1) .

We employ the following notations in our present investigation :

(i) $(m) = 1,2,\ldots, m$;

(ii) $(a_p) = a_1, a_2, \ldots, a_p$.

(iii) $[(a_p)]_n = (a_1)_n (a_2)_n \ldots (a_p)_n$.

(iv) $[(a_{m;i})_{(p_{m;i})}] = (a_1)_{p_1} (a_2)_{p_2} \ldots (a_{i-1})_{p_{i-1}} (a_{i+1})_{(p_{i+1})} \ldots (a_m)_{p_m}$

(v) $[(a_m x_m)^{p_m}] = (a_1 x_1)^{p_1} (a_2 x_2)^{p_2} \ldots (a_m x_m)^{p_m}$

(vi) $[(a_{m;i}\, x_{m;i})^{p_{m;i}}] = (a_1 x_1)^{p_1} (a_2 x_2)^{p_2} \ldots (a_{i-1}\, x_{i-1})^{p_{i-1}} (a_{i+1}\, x_{i+1})^{p_{i+1}} \ldots (a_m x_m)^{p_m}$

2. THE MAIN RESULTS

We shall now derive four theorems on bilateral generating functions .

THEOREM 1, *If*

$$F[x_m), y;t] = \sum_{n=0}^{\infty} S_n^A[(x_m), y]\, t^n, \tag{2.1}$$

then

$$\frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\alpha+\beta)}{(2\pi i)^2}\int_0 (-p)^{\alpha-1}\,(p-1)^{\beta-1}$$

$${}_2F_1\left[\begin{matrix}1-\beta,\ \gamma; & \dfrac{zp}{(p-1)\,(z-1)}\\ \alpha; & \end{matrix}\right] F[\,(x_m),\, y;\, t_p\,]\; dp$$

$$=(1-z)^{\gamma}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\alpha+\beta)_n}\;{}_2F_1\left[\begin{matrix}-n,\ \gamma; & z\\ \alpha; & \end{matrix}\right] S_n^A\,[\,(x_m),\, y\,]\, t^n, \quad (2.2).$$

where the integration is aken over Pochhammer's double-loop C.

Proof . Let us write tp for t in (2.1), multiply it by a constant multiple of $\int_0 (-p)^{\alpha-1}\,(p-1)^{\beta-1}\;{}_2F_1\left[\begin{matrix}1-\beta,\ \gamma; & \dfrac{zp}{(z-1)\,(p-1)}\\ \alpha; & \end{matrix}\right] dp$,

integrate with respect to p along the Pochhammer's double-loop C. Using the known result [13, p. 256]

$$\left(\frac{1}{2\pi i}\right)^2\int_C (-p)^{\alpha-1}\,(p-1)^{\beta-1}\;dp=\frac{1}{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\alpha+\beta)} \tag{2.3}$$

and simplifying, we get the desired result (2.2), on employing a known transformation for ${}_2F_1$. Alternatively, we may put (2.2) in the form :

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\alpha+\beta)_n}\;{}_2F_1\left[\begin{matrix}-n;\gamma; & z\\ \alpha; & \end{matrix}\right] S_n^A\,[\,(x_m),\, y\,)\, t^n$$

$$=(1-z)^{-\gamma}\; F^{2:1;0;1\,\ldots;1}_{1:1;0;1;\ldots;1}\left[\begin{matrix}(\alpha:1,1,r_2,\ldots,r_m)\,,\\ (\alpha+\beta:0,1,r_2,\ldots,r_m),\end{matrix}\right.$$

$$(a:0.\,0,1,\ldots,1):(\gamma:1)\,;-;)\,b_1:1)\,;\ldots;(b_m:1);$$

$$(\alpha:1)\,;-;(c_1:1)\,;\ldots;(c_m:1)\,;$$

$$\frac{z}{z-1}, \nu yt, \mu_1 x_1 t, \mu_2 x_2 t^{r_2}, \ldots, \mu_m x_m t^{r_m} \Bigg], \tag{2.5}$$

in terms of a generalized Lauricella function of Srivastava and Daoust (*Cf.*, *e. g.*, [3, p. 106]).

THEOREM 2, *If*

$$G[(x_m), y;]\, t = \sum_{n=0}^{\infty} S_n^A [(x_m), y]\, t^n, \tag{2.6}$$

then

$$\frac{\Gamma(1-\alpha_1)\Gamma(1-\beta_1)\Gamma(1-\alpha_2)\Gamma(1-\beta_2)\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)}{(2\pi i)^4}$$

$$\int_C \int_{C'} (-p)^{\alpha_1-1} . (p-1)^{\beta_1-1} (-q)^{\alpha_2-1} (q-1)^{\beta_2-1}$$

$${}_2F_1 \left[\begin{matrix} 1-\beta_1, 1-\beta_2 ; \\ \alpha ; \end{matrix} \; \frac{pq\,z}{(z-1)(p-1)(q-1)} \right]$$

$$. G\left[(x_m), y, \frac{pqt}{1-z} \right] dpdq$$

$$= (1-z)^{\alpha_2} \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n}{(\alpha_1+\beta_1)_n (\alpha_2+\beta_2)_n}$$

$${}_2F_1 \left[\begin{matrix} -n, \alpha_2+n ; \\ \alpha_1 ; \end{matrix} \; z \right] S_n^A [(x_m), y]\, t^n, \tag{2.7}$$

where c and c' denote Pochhamner's double-loops .

The proof of Theorem 2 is very similar to that of Theorem 1. Theorem 2 also yields a new bilateral generating relation :

$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n}{(\alpha_1+\beta_1)_n (\alpha_2+\beta_2)_n} \, {}_2F_1\left[\begin{matrix} -n, \alpha_2+n ; \\ \alpha_1 ; \end{matrix} z\right] S_n^A [(x_m), y] \, t^n$$

$$= (1-z)^{-\alpha_2} F^{3:0;1;\dots;1}_{2:1;\,\dots;1}\left[\begin{matrix} (\alpha_1 : 1, 1, r_2, \dots, r_m), \\ (\alpha_1+\beta_1 : 0, 0, r_2, \dots, r_m), \end{matrix}\right.$$

$$(\alpha_2 ; 1, 1, r_2, \dots, r_m), (a : 0, 0, 1, \dots, 1) : -; (b_1 : 1) ; \dots ; (b_m : 1) ;$$

$$(\beta_2+\alpha_2 : 0, 0, r_2, \dots, r_m), \; - : - ; (c_1 : 1) ; \dots ; (c_m : 1) ;$$

$$\left. \frac{z}{z-1}, \frac{\nu y t}{1-z}, \frac{\mu_1 x_1 t}{1-z}, \frac{\mu_2 x_2 t^{r_2}}{(1-z)^{r_2}}, \dots, \frac{\mu_m x_m t^{r_m}}{(1-z)^{r_m}} \right], \tag{2.8}$$

in terms of the aforementioned generalized Lauricella function.

Similar are the proofs of the following theorems :

THEOREM 3. *If*

$$F[(x_m), y ; t] = \sum_{n=0}^{\infty} S_n^A [(x_m), y] \, t^n, \tag{2.9}$$

then

$$\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\alpha+\beta)}{(2\pi i)^2} \int_C (-p)^{\alpha-1} (p-1)^{\beta-1}$$

$$\cdot F_D \left[1-\beta, \alpha_1, \ldots, \alpha_n ; \gamma ; \frac{z_1 p}{(z_1-1)(p-1)}, \ldots, \right.$$

$$\left. \frac{z_n p}{(z_n-1)(p-1)} \right] F_1 [(x_m), y ; t] \, dp$$

$$= (1-z_1)^{\alpha_1} \ldots (1-z_n)^{\alpha_n} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\alpha+\beta)_n}$$

$$F_D [-n, \alpha_1, \ldots, \alpha_n ; \gamma ; z_1, \ldots, z_n] \cdot S_n^A [(x_m), y] t^n \quad (2.10)$$

THEOREM 4. *If*

$$H [(x_m), y ; t] = \sum_{n=0}^{\infty} S_n^A [(x_m), y] t^n , \quad (2.11)$$

then

$$\frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\alpha+\beta) \prod_{i=1}^{n} \left\{ \Gamma(1-\alpha_i)\Gamma(1-\beta_i)\Gamma(\alpha_i+\beta_i) \right\}}{(2\pi i)^{2n+2}}$$

$$\int_{C_1} \cdots \int_{C_{n+1}} (-p)^{\alpha-1} (p-1)^{\beta-1} \left\{ \prod_{i=1}^{n} (-p_i)^{\alpha_i-1} (p_i-1)^{\beta_i-1} \right\}$$

$$\cdot F_A \left[1-\beta, 1-\beta_1, \ldots, 1-\beta_n ; \alpha_1, \ldots, \alpha_n ; \right.$$

$$\frac{pp_1 \; z_1}{(z_1+\ldots+z_n-1)(p-1)(p_1-1)}, \ldots$$

$$\ldots, \frac{pp_n \; z_n}{(z_1+\ldots+z_n-1)(p-1)(p_n-1)}\Bigg]$$

$$H[(x_m), y; \frac{tp \; p_1 \ldots p_n}{(1-z_1 \ldots -z_i)}] \, dp dp_1 \ldots dp_n$$

$$= (1-z_1 \ldots -z_n)^{\alpha} \sum_{n=0}^{\infty} \frac{(\alpha)_n \prod_{i=1}^{n} (\alpha_i)_n}{(\alpha+\beta)_n \prod_{i=1}^{n} (\alpha_i+\beta_i)_n}$$

$$F_A[\alpha+n, -n, \ldots, -n; \alpha_1, \ldots, \alpha_n; z_1, \ldots, z_n] \, S_n^A[(x_m), y] \, t^n, \qquad (2.12)$$

where C_j ($j=1, 2, \ldots, n+1$) denote Pochhammer's double loops.

3. APPLICATIONS

By specializing various parameters in (2.2), we get (see [12])

$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n}{(\alpha_1+\beta_1)_n \, (1+\alpha)_n} \, {}_2F_1[-n, \gamma; \alpha_1; z] \, L_n^{(\alpha)}(x) \, t^n$$

$$= (1-z)^{-\gamma} \; F^{1::0;0;0:0;0;1}_{0::1;0;0:0;1;1} \left[\begin{matrix} \alpha_1 :: -;-;-:, -;-; \gamma; \\ -::\alpha_1+\beta_1;-;-: 1+\alpha; \alpha_1; -; \end{matrix} \; t, -xt, \frac{z}{z-1} \right], \qquad (3.1)$$

which is presumably a new result. Here, and in what follows, $F(x, y, z)$ denotes a general triple hypergeometric function of Srivastava (*Cf.*, *e.g.*, [3, p. 109, Eq. (3.7.4)]).

Applying the usual conditions, from (2.2), we get (see [12])

$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n}{(\alpha_1+\beta_1)_n\,(\beta+1)_n}\ {}_2F_1[-n, \gamma; \alpha_1; z]\ P_n^{(\beta,\ \alpha-n)}(x)\, t^n$$

$$= (1-z)^{-\gamma} F^{1::0;0;0:0;1;1}_{0::0;0;0:0;1;1}\left[\begin{matrix}\alpha_1;:-;-;-:-;1+\alpha+\beta;\gamma;\\ -::\alpha_1+\beta_1;-;-:-;1+\beta;\alpha;\end{matrix}\ t,\ \frac{x-1}{2}t,\ \frac{z}{z-1}\right]. \qquad (3.2)$$

With suitable parameters, (2.5) gives (see [12])

$$\sum_{n=0}^{\infty} \frac{n!\,(\alpha_2)_n}{(\alpha_1+\beta_1)_n\,(\alpha_2+\beta_2)_n\,(1+\alpha)_n}$$

$$P_n^{(\alpha_1-1,\ \alpha_2-\alpha_1)}(1-2z)\ L_n^{(\alpha)}(x)\ t^n$$

$$= (1-z)^{-\alpha}\ F^{2::0;0;0:0;0;0}_{0::2;0;0:0;1;1}$$

$$\left[\begin{matrix}\alpha_1,\alpha_2:: & - & ;-;-:-;-;-;\\ - & ::\alpha_1+\beta_1,\alpha_2+\beta_2 & ;-;-;-;1+\alpha;\alpha_1;\end{matrix}\right.$$

$$\left.\frac{t}{1-z},\ \frac{-xt}{1-z}.\ \frac{-z}{1-z}\right]. \qquad (3.3)$$

Changing F_D to F_1 and $S_n^A[(x_m), y]$ to F_2, (2.10) becomes (see [11])

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n (a)_n (\mu_1 x_1)^n}{n!\,(\alpha+\beta)_n} F_1 [-n, \gamma_1, \gamma_2 ; \alpha ; z_1, z_2]$$

$$\cdot F_2\left[-n, -, b_2 ; 1-a-n, c_2 ; \frac{\nu y}{\mu_1 x_1}, \frac{-\mu_2 x_2}{\mu_1 x_1}\right] t^n$$

$$= (1-z_1)^{-\gamma_1} (1-z_2)^{-\gamma_2} F^{2:1;1;0;0;1}_{2:0;0;0;1;1}\left[\begin{matrix}(-n:0;0;0;1;1); \\ (\alpha+\beta:0;0;1;1\ 1);\end{matrix}\right.$$

$$\begin{matrix}(a:1;..;1):(\gamma_1:1);(\gamma_2:1);-;-; & (b_2:1); \\ (a:1,1,0,0,0): \quad - \quad ; \quad - \quad ;-; & (1-a-n:1);(c_2:1);\end{matrix}$$

$$\left.\frac{z_1}{z_1-1}, \frac{z_2}{z_2-1}, \nu y t, \mu_1 x_1 t, \mu_2 x_1 t, \mu_2 x_2 t\right] \tag{3.4}$$

Changing $S^A_n [(x_m), y]$ to F_A (2.12) reduces to (see [11])

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n [(\alpha_n)]_n (a)_n}{(\alpha+\beta)_n [(\alpha_n+\beta_n)]_n n!} F_A [\alpha+n, -n, .., -n ; \alpha_1, .., \alpha_1 ;$$

$$z_1, .., z_n] F_A\left[-n, -, b_2, .., b_m ; 1-a-n, c_2, .., c_m ;\right.$$

$$\left.\frac{\nu y}{\mu_1 x_1}, \frac{-\mu_2 x_2}{\mu_1 x_1}, \ldots, \frac{-\mu_m x_m}{\mu_1 x_1}\right] (\mu_1 x_1 t)^n$$

$$= (1-z_1 \ldots -z_n)^{-\alpha} F^{n+2:0;..;0;1;..;1}_{n+1:1;..;1;0;1;..;1}\left[\begin{matrix}(\alpha:1;..;1); \\ (\alpha+\beta:0,..,0,1,..,1);\end{matrix}\right.$$

$(\alpha_1:1,0,..,0,1,0,..,0)$;..; $(\alpha_n:0,..,0,1,1,1,0,..,0)$; $(a:0,..,1,..,1)$:
$(\alpha_1+\beta_1:0,..,0,1,1,0,..,0)$;..; $(\alpha_n+\beta_n:0,..,0,1,0,..,0,1)$; — :
– ; .. ; – ; – ; $(b_2 ; 1)$;..; $(b_n : 1)$;
$(\alpha_1 ; 1)$;..; $(\alpha_n : 1)$; – ; $(c_2 : 1)$;..; $(c_n : 1)$; $\frac{z_1}{z_1+..+z_n-1}, ...,$

$$\left.\frac{z_n}{z_1+..+z_n-1}, \frac{\nu y t}{1-z_1-..-z_n}, \frac{\mu_1 x_1 t}{1-z_1-..-z_n}, ..., \frac{\mu_m x_m t}{1-z_1-..-z_n}\right] \tag{3.5}$$

REMARK. An immediate advantage of these theorems and their applications is that we may easily obtain various other known as well as new multi-dimensional generating relations .

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