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SOME INEQUALITIES FOR THE MULTIVARIABLE *H*-FUNCTION

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ABSTRACT

In this paper, we establish five inequalities for the multivariable H-function with the help of certain integrals evaluated by Gupta and Jain [2] and some known inequalities given by Luke [3] . The inequ-alities which were given by Raina and Koul [4] , and Anandani and Singh [1] , are Particular cases of these results .

1. INTRODUCTION AND NOTATIONS

The multivariable H-function which was introduced by Srivastava and Panda ([7], [8]) is an extension of the multivariable G-function. Th s multivariabe H-function includes Fox's H- and Meijer's G-functions of one and two variables, the generalized Lauricella function of Srivastava and Daoust [5], Appell functions, the Whittaker functions, etc . Therefore, the results established in this paper are of a general character .

The multivariable H-function is defined by Srivastava and Panda ([7] , [8]) and is represented in the manner already detailed by Srivastava, Gupta and Goyal [6, p. 251] . Throughout this paper, we also use the shorthand notations as follows :

$$(1.1)\quad \nu=\frac{\lambda}{\delta}\,,\ \theta=\frac{\alpha_P}{\beta_P},\quad \Phi=\frac{\alpha_{P+1}}{\beta_{P+1}}\,,\ \Psi=\prod_{j=1}^{P}\frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)}$$

and

$$(1.2)\quad M_1=\frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}+\delta_j^{(i)}\nu)\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}-\gamma_j^{(i)}\nu)}{\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}-\delta_j^{(i)}\nu)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}+\gamma_j^{(i)}\nu)}$$

$$\frac{\prod_{j=1}^{n}\Gamma(1-a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\nu)}{\prod_{j=n+1}^{p}\Gamma(a_i+\sum_{i=1}^{r}\alpha_j^{(i)}\nu)\prod_{j=1}^{q}\Gamma(1-b_j\sum_{j=1}^{r}\beta_j^{(i)}\nu)}$$

and M_2 represents the expression M_1, where ν has been replaced by $\nu+1/\delta$.

$*H\begin{bmatrix} z_1\\ \vdots\\ z_r\end{bmatrix}$ represents the same expression as for $H\begin{bmatrix} z_1\\ \vdots\\ z_r\end{bmatrix}$ (*see* [6,*p*. 251])

only z_i, m_i, p_i q_i, $(c_j^{(i)},\gamma_j^{(i)})_{1,p_i}$ and $(d_j^{(i)},\delta_j^{(i)})_{1,q_i}$ are replaced by suitable parameters as needed in the next section.

2. MAIN RESULTS

The following inequalities are established for real numbers

$z_1, z_2, \ldots, z_r>0$; $\delta>0$; $\sigma>0$; $\beta_j\geqslant\alpha_j\geqslant 0$ $(j=1,2,\ldots,P)$:

$$(2.1)\quad (\sigma\theta)^{-\lambda}\ *H^{m_i+1,\,n_i+1}_{p_i+1,\,q_i+1}\left[\begin{matrix} z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}(1-\lambda,\delta),\ (c_j^{(i)},\gamma_j^{(i)})_{1,p_i}\\ (1-\lambda,\delta),\ (d_j^{(i)},\delta_j^{(i)})_{1,q_i}\end{matrix}\right]$$

$$< \frac{\Psi}{\Gamma(\sigma)} \; {}^{*}H^{m_i+P+1,\, n_i+1}_{p_i+P+1,\, q_i+P+1} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (1-\lambda,\delta),\ (\beta_j-\lambda,\delta)_{1,P},\ (c_j^{(i)},\gamma_j^{(i)})_{1,p_i} \\ (1-\lambda,\delta),\ (\alpha_j-\lambda,\delta)_{1,P},\ (d_j^{(i)},\delta_j^{(i)})_{1,q_i} \end{matrix} \right]$$

$$< \left\{ 1-\frac{2\sigma\theta}{(\sigma+1)\Phi} \right\} M_1\, \delta^{-1}\, z_i^{-\nu}$$

$$+\ \sigma\theta \left\{ \frac{(\sigma+1)\Phi}{2} \right\}^{(-\lambda+1)} {}^{*}H^{m_i+1,\, n_i+1}_{p_i+1,\, q_i+1}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_j \left\{ \frac{(\sigma+1)\Phi}{2} \right\}^{-\delta} \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i} \\ \\ (1-\lambda,\delta),\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i} \end{matrix} \right],$$

$$\text{(2.2)} \qquad \frac{\theta^{-\lambda}}{\Gamma(\sigma)} \; {}^{*}H^{m_i+1,\, n_i+1}_{p_i+1,\, q_i+1} \left[\begin{matrix} z_1 \\ \vdots \\ z_i\, \theta^{-\delta} \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i} \\ (\sigma-\lambda,\delta),\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i} \end{matrix} \right]$$

$$< \frac{\Psi}{\Gamma(\sigma)} \; {}^{*}H^{m_i+P+1,\, n_i+1}_{p_i+P+1,\, q_i+P+1} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (1-\lambda,\delta),\ (\beta_j-\lambda,\delta)_{1,P} \\ (\sigma-\lambda,\delta),\ (\alpha_j-\lambda,\delta)_{1,P} \end{matrix} \right.$$

$$\left. \begin{matrix} (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i} \\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i} \end{matrix} \right]$$

$$< (1-\theta)\, M_1\, \delta^{-1}\, z_i^{-\nu}$$

$$+\ \frac{\theta}{\Gamma(\sigma)} \; {}^{*}H^{m_i+1,\, n_i+1}_{p_i+1,\, q_i+1} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i} \\ (\sigma-\lambda,\delta),\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i} \end{matrix} \right],$$

$$(2.3)\quad M_1\,\delta^{-1} z_i^{-\nu} - \sigma\,\theta\left(1-\frac{\Phi}{2}\right) M_2\,\delta^{-1} z_i^{-[\nu+(1/\delta)]}$$

$$-\frac{\sigma\,\theta\,\Phi}{2\,\Gamma(\sigma+1)}\ {}^*H^{m_i+1,\,n_i+1}_{p_i+1,\,q_i+1}\left[\begin{matrix} z_1\\ \vdots\\ z_r\end{matrix}\,\middle|\,\begin{matrix}(-\lambda,\delta),\ (c_j^{(i)},\gamma_j^{(i)})_{1,p_i}\\ (\sigma-\lambda,\delta),\ (d_j^{(i)},\delta_j^{(i)})_{1,q_i}\end{matrix}\right]$$

$$< \frac{\Psi}{\Gamma(\sigma)}\ {}^*H^{m_i+P+1,\,n_i+1}_{p_i+P+1,\,q_i+P+1}\left[\begin{matrix} z_1\\ \vdots\\ z_r\end{matrix}\,\middle|\,\begin{matrix}(1-\lambda,\delta),\ (\beta_i-\lambda,\delta)_{1,P},\ (c_j^{(i)},\gamma_j^{(i)})_{1,p_i}\\ (\sigma-\lambda,\delta),\ (\alpha_j-\lambda,\delta)_{1,P},\ (d_j^{(i)},\delta_j^{(i)})_{1,q_i}\end{matrix}\right]$$

$$< M_1\,\delta^{-1} z_i^{-\nu} - \frac{\sigma\theta}{\Gamma(\sigma+1)}\left(\frac{\Phi}{2}\right)^{-(\lambda+1)}$$

$${}^*H^{m_i+1,\,n_i+1}_{p_i+1,\,q_i+1}\left[\begin{matrix} z_1\\ \vdots\\ z_i\left(\frac{\Phi}{2}\right)^{-\delta}\\ \vdots\\ z_r\end{matrix}\,\middle|\,\begin{matrix}(-\lambda,\delta),\ (c_j^{(i)},\gamma_j^{(i)})_{1,p_i}\\ (\sigma-\lambda,\delta),\ (d_j^{(i)},\delta_j^{(i)})_{1,q_i}\end{matrix}\right],$$

$$(2.4)\quad \left(-\frac{1}{\sigma}\right) M_1\,\delta^{-1} z_i^{-\nu} + \left\{\left(\frac{\sigma+1}{\sigma}\right)\left(\frac{\Phi}{\sigma+1}\right)^{-\lambda}\right\}$$

$${}^*H^{m_i+1,\,n_i+1}_{p_i+1,\,q_i+1}\left[\begin{matrix} z_1\\ \vdots\\ z_j\left(\frac{\Phi}{\sigma+1}\right)^{-\delta}\\ \vdots\\ z_r\end{matrix}\,\middle|\,\begin{matrix}(1-\lambda,\delta),\ (c_j^{(i)},\gamma_j^{(i)})_{1,p_i}\\ (1-\lambda,\delta),\ (d_j^{(i)},\delta_j^{(i)})_{1,q_i}\end{matrix}\right]$$

$$<\Gamma(\sigma)\,\Psi\ {}^*H^{m_i+P+1,\,n_i+1}_{p_i+P+2,\,q_i+P+1}\left[\begin{matrix} z_1\\ \vdots\\ z_r\end{matrix}\,\middle|\,\begin{matrix}(1-\lambda,\delta),\ (c_j^{(i)},\gamma_j^{(i)})_{1,p_i},\\ (1-\lambda,\delta),\ (\alpha_j-\lambda,\delta)_{1,P},\end{matrix}\right.$$

$$\left.\begin{matrix} (\sigma-\lambda,\delta),\ (\beta_j-\lambda,\delta)_{1,P} \\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i} \end{matrix}\right]$$

$$< \left(1-\frac{\theta}{\Phi}\right) M_1\, \delta^{-1}\, z_1^{-\nu}$$

$$+\frac{\theta}{\Phi}\left(\frac{\Phi}{\sigma}\right)^{-\delta} {}^{*}H^{m_i+1,\, n_i+1}_{p_i+1,\, q_i+1}\left[\begin{matrix} z_1 \\ \vdots \\ z_i\left(\frac{\Phi}{\sigma}\right)^{-\delta} \\ \vdots \\ z_r \end{matrix}\ \middle|\ \begin{matrix} (1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i} \\ (1-\lambda,\delta),\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i} \end{matrix}\right],$$

$$(2.5)\qquad \theta^{-\lambda}\ {}^{*}H^{m_i,\, n_i+1}_{p_i+1,\, q_i}\left[\begin{matrix} z_1 \\ \vdots \\ z_i\,\theta^{-\sigma} \\ \vdots \\ z_r \end{matrix}\ \middle|\ \begin{matrix} (1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i} \\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i} \end{matrix}\right]$$

$$< \Psi\ {}^{*}H^{m_i+P,\, n_i+1}_{p_i+P+1,\, q_i+P}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\ \middle|\ \begin{matrix} (1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i},\ (\beta_j-\lambda,\delta)_{1,P} \\ (\alpha_j-\lambda,\delta)_{1,P},\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i} \end{matrix}\right]$$

$$< (1-\theta)\, M_1\, \delta^{-1}\, z_i^{-\nu} + \theta\ {}^{*}H^{m_i,\, n_i+1}_{p_i+1,\, q_i}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\ \middle|\ \begin{matrix} (1-\lambda,\delta),\ (c_j^{(i)},\ \gamma_j^{(i)})_{1,p_i} \\ (d_j^{(i)},\ \delta_j^{(i)})_{1,q_i} \end{matrix}\right]$$

Proofs : To prove (2.1), we replace z by $z_i t$ in (4.10) of Luke [3,p.53], multiply throughout by

$$t^{\lambda-1}\ H\left[\begin{matrix} z_1 \\ \vdots \\ z_i\, t^{\delta} \\ \vdots \\ z_r \end{matrix}\right]$$

and integrate with respect to t between 0 to ∞. Now, evaluating the integrals involved on the lines of (5.1) of Gupta and Jain [2,*p*. 601] , we get the result (2.1) .

Proceeding on similar lines, the inequalities (2 2) , (2.3) , (2.4) and (2.5) can also be established, using the results of Luke [3, pp. 55 and 57, Equations (4.20), (4.22) , (5.1) and (5.5)] .

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