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COMMENTS ON ISTRATESCU'S CONVEX CONTRACTION MAPS AND MAPS WITH CONVEX DIMINISHING DIAMETERS

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In [3] and [4], V. I. Istratescu considered several classes of maps related to contraction maps by introducing convexity conditions with respect to the iterates of maps, and obtained a number or fixed point theorems on such maps. However, we notice that some of his results are closely related to our previous work ([5], [6]) . In fact, in the present paper, we show that some of Istratescu's key results are obvious consequences of our previous results in [5] and [6] . Further, we obtain generalizations of some of his results .

Let f be a selfmap of a metric space (X, d) . Let $0(x)$ be the orbit of $x \in X$ under f . A point $x \in X$ is said to be regular if $0(x)$ has finite diameter . we denote

$\delta(x, y) = diqm \{ 0(x) \cup 0(y) \}$ *if* x and y are regular .

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Theorem 1 ([6], Corollary to Theorem 1) . *Let f be a continuous selfmap of a complete metric space X . Suppose that*

(i) *there exists a regular point $x \in X$, and*

(ii) *for fixed integers p and q, given $\epsilon > 0$ there exist an ϵ_0, $0 < \epsilon_0 < \epsilon$, and $\delta_0 > 0$ such that for each $x, y \in X$.*

$$\epsilon \leqslant \delta (x,y) < \epsilon + \delta_0 \text{ implies } d (f^p x, f^q y) \leqslant \epsilon_0 .$$

Then f has unique fixed potnt $z \in X$ and $f^n \to z$ for each regular $x \epsilon X$.

We now show that Theorem 1 includes Theorems 1.2, 1.5, 2.3 and 2.4 of Istratescu [3] :

Let f be *a* continuous selfmap of a complete metric space X satisfying one of the following conditions :

(1) there exists $a_0, a_1 , \ldots , a_{n-1} \in (0,1)$,

$a_0 + \ldots + a_{n-1} < 1$, such that for all $x, y \in X$,

$$d (f^n x, f^n y) \leqslant a_0 d (x,y) + a_1 d (fx, fy) + \ldots + a_{n-1} d (f^{n-1}x, f^{n-1}y)$$

(2) there exist $c_0 , c_1 , a_1 , a_2 , b_1, b_2 \in (0,1)$,

$c_0 + c_1 + a_1 + a_2 + b_1 + b_2 < 1$, such that for all $x, y \in X$,

$$d (f^2x, f^2y) \leqslant c_0 d (x,y) + c_1 d (fx, fy) + a_1 d (x, fx)$$
$$+ a_2 d (fx, f^2x) + b_1 d (y, fy) + b_2 d (fy, f^2y) .$$

Note that each of (1) and (2) (with $n=2$) implies

$$d (f^n x, f^n y) \leqslant c \, diam\{ x, fx , \ldots , f^n x, y, fy , \ldots, f^n y \}$$

where $c \in (0,1)$ and whence the orbit $0(x)$ of any $x \in X$ is bouuded (e. g. see B. Fisher [2]) . Therefore, we have

$$d (f^n x, f^n y) \leqslant c \delta (x,y) ,$$

and by the argument in [6], all the hypotheses of Theorem 1 are satisfied by putting $p = q = n$. Hence f has a unique fixed point $z \in X$ and $f^n x \to z$ for all $x \in X$.

Since obvious modifications of Theorem 1 hold for a generalized metric space (in which the distance can take on the value $+\infty$) and a Hausdorff uniform s space, Istrateseu's Theorems 4.2 [3] and 4.5 [4] can also be extended .

Theorem 2 [6] . *Let f be a continuous selfmap of a metric space X. Suppose that for fixed integers p and q,*

$$d (f^p x , f^q y) < \delta (x, y)$$

for each $x, y \in X$ $x \neq y$. Then f has a unique fixed point . Furthermore, for any $\alpha \in (0,1)$ there exists a metric ρ on X, relative to which f is a Banach contraction with Lipschitz constant α .

We now show that Theorem 2 includes Theorem 1.7 of Istratescu [3]:

Let f be a continuous selfmap of a compact metric space X such that there exist a_0, $a_1 \in (0,1)$, $a_0 + a_1 = 1$, and for all $x, y \in X$, $x \neq y$,

$$d (f^2 x, f^2 y) < a_0 d(x, y) + a_1 d(fx, fy) .$$

Then the hypotheses of Theorem 2 are clearly satisfied .

Theorem 3. *Let f be a continuous selfmap of a metric space X and $F : X \to R$ a continuous function satisfying :*

(i) $F (fx) \leqslant F (x)$ *for all* $x \in X$,

(ii) *if* $x \neq fx$, *then* $F (fx) < F (x)$, *and*

(iii) *for some* $x \in X$, *the sequence* $\{ f^n x \}$ *has a cluster point* $z \in X$. *Then z is a fixct point of f.*

Proof . By (i), the sequence $\{ F (f^n x) \}$ is nonincreasing. Set $z = lim_k f^{n_k} x$. By (i), we have

$$F(z) = lim_k F[f^{n_k}x] = lim_n F(f^n x)$$

$$= lim_k F[f^{n_k+1}x] = F(fz),$$

contradicting (ii) unless $z = fz$.

In the case where $F(x) = d(x, fx)$, Theorem 1 extends a well-known result of Edelstein.

We show that Theorem 1.8 of Istratescu [3] follows from Theorem 3 :

Let $f: X \to X$ satisfying

(3) $d(f^2x, f^2y) \leqslant ad(fx, fy) + bd(x, y)$, $a, b, \in (0,1)$, $a+b<1$, for all $x, y \in X$. Since $a < 1 - b$, we have

$$d(f^2x, f^3x) \leqslant bd(x, fx) + (1-b)\, d(fx, f^2x).$$

By putting

$$F(x) = bd(x, fx) + d(fx, f^2x),$$

we have $F(fx) \leqslant F(x)$. Suppose $F(fx) = F(x)$. Then

$$bd(fx, f^2x) + d(f^2x, f^3x) = bd(x, fx) + d(fx, f^2x),$$

and

(4) $d(f^2x, f^3x) = bd(x, fx) + (1-b)\, d(fx, f^2x)$.

If $fx \neq f^2x$, then we have $d(f^2x, f^3x) > bd(x, fx) + ad(fx, f^2x)$, contradicting (3). Therefore $fx = f^2x$, which implies $f^2x = f^3x$ and, from (4) since $b > 0$, $x = fx$. Therefore (ii) is satisfied. The uniqueness of the fixed point is clear.

Corollary 1. *Let f a continuous selfmap of a metric space* (X, d) *and* $F: X \times X \to \mathbb{R}$ *a continuous function such that*

(i) $F(fx, fy) \leqslant F(x,y)$ *for all* $x,y \in X$,

(ii) *if* $x \neq fx$, *then* $F(fx, f^2x) < F(x, fx)$, *and*

(iii) *for each* $x \in X$, *the sequence* $\{f^n x\}$ *has a cluster point* $z \in X$. *Then* z *is a fixed point of* f,

Proof. Replace $F(x)$ in Theorem 3 by $F(x, xf)$.

In [1], W. Cheney and A. A. Goldstein otained the above corollary for the case $F(x, y) = d(x, y)$. In this case, for each $x \in X$, $\{f^n x\}$ converges to a fixed point of f.

Theorem 1.6 of [4] follows from the corollary by defining

$$F(x, y) = (1-s)\, d(x, y) + d(fx, fy).$$

Theorem 4 ([5], proposition 1). *Let* X *be a complete metric space and* $\{A_n\}$ *a sequence of nonempty subsets of* X *such that*

$A_{n+1} \subset A_n$ *for each* n. *Let* $d_n = diam(A_n)$ *satisfy the following* :

For aay $\epsilon > 0$, *there exists a* $\delta > 0$ *snch that*

$$\epsilon \leqslant d_n < \epsilon + \delta \text{ implies } d_{n+1} < \epsilon .$$

Then $A = \bigcap_n A_n$ *is a singleton* .

Corollary 2. ([3], Theorem 3.3). *Let* f *be a selfmap of a complete metric space* X *such that for any bounded subset* M *of* X, *there exist an* $a \in [0,1]$ *and an integer* $n = n(M)$ *such that*

$$diam\,(f^n(M)) \leqslant a\ diam\,(M).$$

If X *is bounded, then* f *has a unique fixed point* .

Proof. *Let* $A_1 = X$, $A_2 = f^{n(A_1)} A_1, \ldots, A_m = f^{n(A_{m-1})} A_{m-1}, \ldots$, and $d_n = diam\,(A_n)$. Since $d_n \leqslant a d_{n-1}$, by Theorem 4, $A = \bigcap_n \bar{A}_n$ is a singleton, which is the unique fixed point of f.

Since an obvious modification of Theorem 4 holds for a Hausdorff uniform space, Istratescu's Theorem 4.8 [4] can also be extended.

Theorem 5. *Let f be a selfmap of a topological space X and $F : X \to \mathbb{R}$ a l. s. c. function. Suppose that*

(i) *if $x \neq fx$, $x \in X$, then $F(fx) < F(x)$, and*

(ii) *the range of f is relatively countably compact. Then f has a fixed point.*

Proof. *Let* $A := \overline{fX}$. Since A is countably compact and F is *l. s. c.* on A, F attains *its* infimum on A; that is, there exists a $u \in A$ such that $F(u) \leqslant F(x)$ for all $x \in A$. Suppose that $u \neq fu$. Then $F(fu) < F(u)$, which is a contradiction.

To prove Theorem 4.12 of [4], set $F(x) = g(x, fx)$.

Since continuous densifying selfmaps of a complete metric space have relatively compact orbits whenever the orbits are bounded, Theorem 5 can also be applied to sueh maps.

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