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# STABILITY OF STRATIFIED FLUID IN POROUS MEDIUM IN THE PRESENCE OF VARIABLE MAGNETIC FIELD AND UNIFORM ROTATION

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## ABSTRACT

A study has been made of the stability of stratified fluid in porous medium in the presence of variable horizontal magnetic field and uniform rotation. Both the density and the magnetic field are assumed to be exponentially varying. For stable stratification, the system is found to be stable for disturbances of all wave-numbers. The magnetic field succeeds in stabilizing potentially unstable stratification for small wave-length perturbations $k > (g\beta)^{1/2} \sec\theta/V$ which were unstable in its absence. The long wave-length perturbations satisfying $k^2 < \frac{g\beta \sec^2\theta}{V^2} - \frac{m^2\pi^2}{d^2}$ remain unstable and are not stabilized by magnetic field.

## 1. INTRODUCTION

Chandrasekhar [1] has given a detailed account of the Rayleigh-Taylor instability under varying assumptions of hydrodynamics and hydromagnetics . The magnetic field has been considered to be constant and uni-directional throughout . Gupta [2] has studied the stability of a horizontal layer of a perfectly conducting flnid, wtth continuous density and viscosity stratification in the presence of a horizontal magnetic field ( constant as well as variable ) . When a fluid slowly permeates a porous material, the gross effect is represented by the usual Darcy's law . As a result of this, the usual viscous term in the equations of fluid motion is replaced by the resistance term ( $\mu/k_1$ ) **v**, where $\mu$ is the viscosity of the fluid, $k_1$ the permeability of the medium and **v** the velocity of the fluid, calculated from Darcy's law . The Rayleigh instability of a thermal boundary layer in flow through a porous medium has been considered by Woodirg [5] . Sharma [4] has studied the effect of uniform magnetic field and uniform rotation on the stability of two superposed fluids in porous medium .

Usually the magnetic field has a stabilizing effect on the instability . However, Kent [3] has studied the effect of a horizontal magnetic field which varies in the vertical direction on the stability of parallel flows and has shown that system is unstable under certain conditions, while in the absence of magnetic field the system is known to be stable . In the interior of the Earth, the magnetic field may be variable and may altogether alter the nature of the instability . The Coriolis force also plays an important role on the stability of geophysical phenomena .

The present paper, therefore , deals with the stability of stratified fluid in porous medium in the presence of variable magnetic field and uniform rotation .

## 2. Perturbation Equations

Here we study the Rayleigh-Taylor instability of a fluid with continuous density stratification in porous medium in the presence of rotation and a variable horizontal magnetic field. The fluid is considered to be heterogeneous, incompressible, infinitely conducting and infinitely extending so that the free surface is almost horizontal.

The fluid is acted on by gravity $\mathbf{g}(0,0,-g)$, a uniform rotation $\vec{\Omega}(0,0,\Omega)$ and a variable horizontal magnetic field $\mathbf{H}(H_0(z), 0,0)$.

Let $\rho$, $p$ and $\mathbf{v}(u, v, w)$ denote respectively the density, pressure and velocity of the fluid; $\mu_e$ is the magnetic permeabiliiy and $\epsilon$ is the medium porosity. Then the hydromagnetic equations relevant to the problem are

$$\frac{\rho}{\epsilon}\left[\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{\epsilon}(\mathbf{v}.\nabla)\mathbf{v}\right]=-\nabla p+\frac{\mu_e}{4\pi}(\nabla\times H)\times H+\rho \mathbf{g}$$

$$-\frac{\mu}{k_1}\mathbf{v}+\frac{2\rho}{\epsilon}(\mathbf{v}\times\vec{\Omega}), \tag{1}$$

$$\nabla\cdot\mathbf{v}=0, \tag{2}$$

$$\epsilon\frac{\partial \mathbf{H}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{H}), \tag{3}$$

$$\nabla\cdot\mathbf{H}=0. \tag{4}$$

Since the density of a particle moving with the fluid remains unchanged, we have

$$\epsilon\frac{\partial\rho}{\partial t}+(\mathbf{v}\cdot\nabla)\rho=0, \tag{5}$$

Let $\delta\rho$, $\delta p$, $\mathbf{v}(u, v, w)$ and $\mathbf{h}(h_x, h_y, h_z)$ denote the perturbations in density $\rho$, pressure $p$, velocity $(0, 0, 0)$ and magnetic field $\mathbf{H}(H_0(z), 0,0)$ respectively. Then the linearized hydromagnetic perturbation equations are

$$\frac{\rho}{\epsilon}\frac{\partial \mathbf{v}}{\partial t}=-\nabla\delta p+\mathbf{g}\,\delta\rho-\frac{\mu}{k_1}\mathbf{v}+\frac{2\rho}{\epsilon}(\mathbf{v}\times\vec{\Omega})$$

$$+\frac{\mu_e}{4\pi}\left[(\nabla\times\mathbf{h})\times\mathbf{H}+(\nabla\times\mathbf{H})\times\mathbf{h}\right], \tag{6}$$

$$\nabla\cdot\mathbf{v}=0, \tag{7}$$

$$\epsilon\frac{\partial\mathbf{h}}{\partial t}\ \nabla\times(\mathbf{v}\times\mathbf{H}), \tag{8}$$

$$\nabla\cdot\mathbf{h}=0, \tag{9}$$

$$\epsilon\frac{\partial}{\partial t}\ \delta\rho=-\omega\frac{d\rho}{dz}\ . \tag{10}$$

Analyzing the disturbances into normal modes, we seek solutions whose dependence on $x$,y and $t$ is given by

$$exp\,(\,ik_x x+ik_y y+nt\,)\,, \tag{11}$$

where $k_x$, $k_y$ are the horizontal wave numbers, $k=(k_x^2+k_y^2)^{1/2}$ and $n$ is the growth rate .

Equations (6)-(10), using expression (11), become

$$\frac{\rho}{\epsilon}\ n\,u=-\,ik_x\,\delta p+\frac{\mu_e}{4\pi}\,h_z\,DH_0-\frac{\mu}{k_1}\,u+\frac{2\rho\Omega}{\epsilon}\,v\,, \tag{12}$$

$$\frac{\rho}{\epsilon}\ n\,v=-\,i\,k_y\,\delta p+\frac{\mu_e H_0}{4\pi}\,(\,ik_x h_y-ik_y h_x\,)-\frac{\mu}{k_1}\,v-\frac{2\rho\Omega}{\epsilon}\,u\,, \tag{13}$$

$$\frac{\rho}{\epsilon}\ n\omega=-\,D\,pd+\frac{\mu_e H_0}{4\pi}\,(\,ik_x h_z-Dh_x-h_x\frac{DH_0}{H_0}-\frac{\mu}{k_1}\,\omega+\frac{g}{n\epsilon}\,(\,D\rho\,)\,\omega\,, \tag{14}$$

$$ik_x u+ik_y v+Dw=0\,, \tag{15}$$

$$ik_x h_x+ik_y h_y+Dh_z=0\,, \tag{16}$$

$$n\epsilon\mathbf{h}=ik_x H_0\,\mathbf{v}-\omega\,(\,DH_0\,)\,\mathbf{I}_z\,, \tag{17}$$

where $\mathbf{I}_x$ ( 1,0,0 ) is the unit vector in the direction of $x$–axis and $D = d/dz$ .

Multiplying eqs. (12) and (13) by $-ik_x$, $-ik_y$ respectively , adding and substituting for $h_x, h_y, h_z$ ; we get

$$\frac{\rho}{\epsilon} n D\omega = - k^2 \delta p - \frac{\mu}{k_1} D\omega - \frac{2\rho\Omega}{\epsilon} \zeta + \frac{k_x k_y}{\epsilon} \frac{\mu_e H_0^2}{4\pi n} \zeta$$

$$+ \frac{\mu_e H_0 k^2}{4\pi n \epsilon} \omega DH_0 , \qquad (18)$$

where $\zeta ( = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} )$ , the $z$-component of vorticity, is given by

$$\zeta = \frac{2\Omega D\omega}{n + \frac{\nu\epsilon}{k_1} + \frac{k_x^2 V^2}{n}} , \qquad (19)$$

which is obtained by multiplying eqs . (12) and (13) by $-ik_y$ and $+ik_x$ respectively , adding and substituting for $h_x$, $h_y$, $h_z$ .

$V^2 = \frac{\mu_e H^2}{4\pi\rho}$ is the square of the Alfvén velocity .

Eliminating $\delta p$ between eqs , (14) and (18), using (19) and the relation

$$ik^2 u = - ( k_x D\omega + k_y \zeta ) = - \left( k_x + \frac{2k_y \Omega n}{n^2 + \frac{\nu\epsilon}{k_1} n + k_x^2 V^2} \right) D\omega , \qquad (20)$$

we get after simplification

$$\left[ 1 + \frac{4\Omega^2}{n^2 + \frac{\nu\epsilon n}{k_1} + k_x^2 V^2} \right] D ( \rho Dw ) - k^2 \rho\omega + \frac{\mu\epsilon}{k_1 n} ( D^2 - k^2 ) \omega$$

$$+\frac{\mu_e k_x^2}{4\pi n^2}\left[ D( H_0^2 D\omega ) - k^2 H_0^2 \omega \right] = -\frac{gk^2}{n^2}( D\rho )\, \omega . \tag{21}$$

Equation (21) is the general equation formulating the effect of uniform rotaiton and a variable horizontal magnetic field on the stability of stratified fluid in a porous medium. For the case of a uniform horizontal magnetic field, eq. (21) reduces to the result ( Sharma [4] ).

## 3. The Case of Exponentially Varying Density and Magnetic Field

Let us assume the stratifications in density and magnetic field of the form

$$\rho = \rho_1 e^{\beta z} , \quad H_0^2 = H_1^2 e^{\beta z} , \tag{22}$$

where $\rho_1$, $H_1$ and $\beta$ are constants and so

$$V^2 = \frac{\mu_e H_0^2}{4\pi\rho} = \frac{\mu_e H_1^2}{4\pi\rho_1} . \tag{23}$$

Using the stratification of the form (22), eq. (21) transforms to

$$\left[ 1 + \frac{k_x^2 V^2}{n^2} + \frac{\nu\epsilon}{k_1 n} + \frac{4\Omega^2}{n^2 + \frac{\nu\epsilon}{k_1} n + k_x^2 V^2} \right] D^2\omega$$

$$+ \left[ 1 + \frac{k_x^2 V^2}{n^2} + \frac{4\Omega^2}{n^2 + \frac{\nu\epsilon}{k_1} n + k_x^2 V^2} \right] \beta D\omega$$

$$- \left[ 1 + \frac{k_x^2 V^2}{n^2} + \frac{\nu\epsilon}{k_1 n} - \frac{g\beta}{n^2} \right] k^2\omega = 0 . \tag{24}$$

The general solution of eq. (24) is

$$\omega = A_1 e^{q_1 z} + A_2 e^{q_2 z} \quad , \tag{25}$$

where $A_1$ , $A_2$ are two arbitrary constants and $q_1$ , $q_2$ are the roots of the equation

$$\left[1 + \frac{k_x^2 V^2}{n^2} + \frac{\nu\epsilon}{k_1 n} + \frac{4\Omega^2}{n^2 + \frac{\nu\epsilon}{k_1} n + k_x^2 V^2}\right] q^2$$

$$+ \left[1 + \frac{k_x^2 V^2}{n^2} + \frac{4\Omega^2}{n^2 + \frac{\nu\epsilon}{k_1} n + k_x^2 V^2}\right] \beta q$$

$$- \left[1 + \frac{k_x^2 V^2}{n^2} + \frac{\nu\epsilon}{k_1 n} - \frac{g\beta}{n^2}\right] k^2 = 0 \, . \tag{26}$$

If the fluid is supposed to be confined between two rigid planes at $z = 0$ and $z = d$, then the vanishing of $w$ at $z = 0$ is satisfied by the choice

$$\omega = A \left( e^{q_1 z} - e^{q_2 z} \right) , \tag{27}$$

while the vanishing of $w$ at $z = d$ reqires

$$exp \left( q_1 - q_2 \right) d = 1 \, , \tag{28}$$

which implies that

$$\left( q_1 - q_2 \right) d = 2 i m \pi \, , \tag{29}$$

where $m$ is an integer .

Equation (26) gives

$$q_{1,2}=\Bigg[-\frac{\beta}{2}\left(1+\frac{k_x^2V^2}{n^2}+\frac{4\Omega^2}{n^2+\frac{\nu\epsilon}{k_1}n+k_x^2V^2}\right)$$

$$\pm\tfrac{1}{2}\Bigg\{\beta^2\left(1+\frac{k_x^2V^2}{n^2}+\frac{4\Omega^2}{n^2+\frac{\nu\epsilon}{k_1}n+k_x^2V^2}\right)^2$$

$$+4k^2\left(1+\frac{k_x^2V^2}{n^2}+\frac{\nu\epsilon}{k_1n}+\frac{4\Omega^2}{n^2+\frac{\nu\epsilon}{k_1}n+k_x^2V^2}\right)\times$$

$$\times\left(1+\frac{k_x^2V^2}{n^2}+\frac{\nu\epsilon}{k_1n}-\frac{g\beta}{n^2}\right)\Bigg\}^{1/2}\Bigg].$$

$$\cdot\left(1+\frac{k_x^2V^2}{n^2}+\frac{\nu\epsilon}{k_1n}+\frac{4\Omega^2}{n^2+\frac{\nu\epsilon}{k_1}n+k_x^2V^2}\right)^{-1}. \qquad (30)$$

Inserting the values of $q_1$, $q_2$ from (30) in (29) and simplifying, we obtain

$$A_8n^8+A_7n^7+A_6n^6+A_5n^5+A_4n^4+A_3n^3+A_2n^2+A_1n+A_0=0\,, \qquad (31)$$

Equation (31) is the dispersion relation studying the effect of rotation and the variable ( exponential ) horizontal magnetic field on the stability of stratified ( exponentially varying density ) fluid in porous medium .

If $\beta<0$ ( stable stratification ) , eq . (31) does not admit of any positive root of $n$ and the system is always stable for disturbances of all wave unmbers .

If $\beta > 0$ ( unstable stratification ) and if

$$V^2 < \frac{g\beta k^2}{\left(k^2+\frac{m^2\pi^2}{d^2}\right)k_x^2}, \tag{32}$$

the constant term in eq. (31) is negative, Equation (31) therefore has at least one positive real root and so the system is unstable for all wave-numbers satisfying the inequality

$$k^2 < \frac{g\beta\, sec^2\theta}{V^2} - \frac{m^2\pi^2}{d^2}, \tag{33}$$

where $\theta$ is the angle between $k_x$ and $k$ ( *i.e.* $k_x = k \cos\theta$ ).

If $\beta > 0$ ( unstable stratification ) but $k_x^2 V^2 > g\beta$, eq. (31) admits of no positive root and so the system is stable.

Thus for unstable stratification but magnetic field such that $V^2 < g\beta\, sec^2\theta / ( k^2 + \frac{m^2\pi^2}{d^2} )$, the system is unstable for all wave-numbers satisfying $k^2 < \frac{g\beta\, sec^2\theta}{V^2} - \frac{m^2\pi^2}{d^2}$.

## 4. RESULTS

The results of the paper are presented here.

( i ) The dispersion relation studying the effect of rotation and the variable ( exponential ) horizontal magnetic field on the stability of stratified ( exponentially varying density ) fluid in porous medium has been obtained.

( ii ) For stable stratification, the system is fonud to be stable for disturbances of all wave-numbers.

(iii) The magnetic field stabilizes potentially unstable stratification for small wave-length perturbations $k > (g\beta)^{1/2} \sec\theta / V$ which were unstable in its absence .

( iv ) The long wave-length perturbations satisfying

$k^2 < \frac{g\beta \sec^2\theta}{V^2} - \frac{m^2\pi^2}{d^2}$ remain unstable and are not stabilized by magnetic field .

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## APPENDIX

The coefficients $A_0 - A_8$ in eq. (31) are given below .

$$A_8 = \beta^2 + 4k^2 + \frac{4m^2\pi^2}{d^2} ,$$

$$A_7 = a\,( 2\beta^2 + 16k^2 + \frac{16m^2\pi^2}{d^2} ),$$

$$A_6 = \beta^2 ( 4k_x V^2 + a^2 ) + \frac{16m^2\pi^2}{d^2} ( k_x^2 V^2 + a^2 )$$

$$+ 4k^2 \{ 4\Omega^2 + 6a^2 + ( 4k_x^2 V^2 - g\beta ) \} ,$$

$$A_5 = 2a\beta^2 ( 3k_x^2 V^2 + 4\Omega^2 ) + \frac{16m^2\pi^2}{d^2} a ( 3k_x^2 V^2 + 4\Omega^2 + a^2 )$$

$$+ 4ak^2 \{ 9k_x^2 V^2 + 8\Omega^2 + 4a^2 + 3 ( k_x^2 V^2 - g\beta ) \} ,$$

$$A_4 = 2\beta^2 ( 3k_x^4 V^4 + k_x^2 V^2 a^2 + 8\Omega^4 + 4\Omega^2 ) + 4k^2 \{ 3k_x^2 V^2 ( 2k_x^2 V^2 - g\beta)$$

$$+ a^4 + 3a^2 ( 4k_x^2 V^2 - g\beta ) + 4\Omega^2 ( 2k_x^2 V^2 - g\beta ] + 4\Omega^2 a^2 \}$$

$$+ \frac{4m^2\pi^2}{d^2} \{ 2k_x^2 V^2 ( 3k_x^2 V^2 + 4a^2 ) + ( 4\Omega^2 + a^2 )^2 + 2\Omega^2 + 2a^2 \} ,$$

$$A_3 = 2a\beta^2 k_x^2 V^2 ( 3k_x^2 V^2 + 4\Omega^2 ) + 4k^2 \{ 6ak_x^2 V^2 ( 2k_x^2 V^2 - g\beta )$$

$$+ 8a\Omega^2 k_x^2 V^2 + a_3 ( 4k_x^2 V^2 - g\beta) \} + \frac{16m^2\pi^2}{d^2} ak_x^2 V^2 (3k_x^2 V^2 + 4\Omega^2 + a^2),$$

$$A_2 = \beta^2 k_x^2 V^2 ( 4k_x^4 V^4 + a k_x^2 V^2 + 8\Omega^2 )$$

$$+ \frac{8m^2\pi^2}{d^2} \{ 2k_x^6 V^6 + 2a^2 k_x^4 V^4 + k_x^2 V^2 ( 4\Omega^2 + a^2 ) \}$$

$$+ 4k^2 \{ k_x^4 V^4 ( k_x^2 V^2 + 3a^2 ) + k_x^2 V^2 ( k_x^2 V^2 - g\beta ) ( 3k_x^2 V^2 + 3a^2 + 4\Omega^2 ) \},$$

$$A_1 = 2a k_x^6 V^6 ( \beta^2 + 2k^2 ) + 12 a k_x^4 V^4 ( k_x^2 V^2 - g\beta ) + \frac{16 a k^6 V^6 m^2 \pi^2}{d^2},$$

$$A_0 = 4 k_x^6 V^6 \{ k^2 ( k_x^2 V^2 - g\beta ) + \frac{m^2\pi^2}{d^2} k_x^2 V^2 ,$$

and

$$a = \nu\epsilon / k_1 .$$