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# ON THE DEGREE OF APPROXIMATION OF CONTINUOUS FUNCTIONS

*By*

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## 1. INTRODUCTION

Let $f$ be $2\pi$–periodic and L–integrable over $[0, 2\pi]$, and let the Fourier series associated with $f$ at a point $x$ be

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} ( a_n \cos nx + b_n \sin nx )$$

Suppose that a function $f$ is defined on a closed interval $I$. If for some $\alpha$, $0 < \alpha \leqslant 1$, there is a constant $M$ such that

$$(1{\cdot}1) \quad | f(x+h) - f(x) | \leqslant M \, | h |^{\alpha} \quad (x, x+h \in I),$$

we say that $f$ belongs to the class Lip $\alpha$ on $I$ with constant $M$ and write

$$(1.2) \qquad f \in Lip(\alpha, M, I) .$$

Hence onward, I will denote the closed interval $[0, 2\pi]$.

Let $\sum_{n=0}^{\infty} c_n$ be a given series and let $s_n = c_0 + c_1 + \ldots + c_n$.

Then the $(E, q)$ $(q > 0)$–means or $\{s_n\}$ are defined by( [4], $p$.180 )

$$(1.3) \qquad (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k$$

Also Borel's exponential mean of $\{ s_n \}$ is given by ( [ 4 ], *p*. 182 )

$$(1.4) \quad e^{-p} \sum_{n=0}^{\infty} (p_n/n\,!)\, s_n \quad (p > 0).$$

The $(E, q)$ $(q > 0)$ and Borel's exponential means are reqular ( [4], *p*. 179 and *p*. 80 ) . Throughout, we write

$$(1.5) \quad \phi_x(t) = \tfrac{1}{2}\{ f(x+t) + f(x-t) - 2f(x) \}$$

$$(1.6) \quad s_n(x) = \tfrac{1}{2} a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx)$$

$$(1{,}7) \quad P(q, t, n) = (1+q)^{-n} (1 + q^2 + 2q \cos t)^{n/2}$$

$$(1.8) \quad Q(q, t) = \tan^{-1} \{ \sin t / (q + \cos t) \}$$

$$(1.9) \quad A = 2q(\pi(1+q))^{-2}.$$

We also write $t(n, q, x)$ and $T_{\bullet}(p, x)$ for $(E, q)$ $(q > 0)$ and Borel's exponential means of the sequence $\{ s_n(x) \}$ , respectively .

It has been proved in [2] and ]3] ( also see [6] ) that the degree of approximation of $f \in Lip(\alpha, M, I)$ $(0 < \alpha \leqslant 1)$ by Borel's exponential and $(E, q)$ $(q > 0)$ means are $O(p^{-\alpha/2}$ and $O(n^{-\alpha/2})$ respectively, none of which is of the Jackson order . However, Chiu and Holland ( see Holland [5], *p*. 370 ) proved that, for $0 < \alpha < 1$, one can get the Jackson order under some additional conditions on $f$. In this paper, we assume different type of additional conditions on $f \in Lip(\alpha, M, I)$, $0 < \alpha \leqslant 1$, to get the Jackson order by these means. Precisely, we prove the following theorems :

**THEOREM 1.** *Let f be* $2\pi$*-periodic and led* $f \in Lip(\alpha, M, I)$ . *If*

(1.10) $t^{-1}(\phi_x(t) + Mt^{\alpha})$ *is a non-increasing function of t in* $(0,\delta)$, *where* $0 < \delta < \pi/4$ , *then the degree of approximation of f, by* $(E, q)$ $(q > 0)$ *means of its Fourier series, is given by*

(1.11) $$\sup_{0 \leqslant x \leqslant 2\pi} | f(x) - t(n, q, x) | = O(n^{-\alpha}) .$$

**THEOREM 2.** *Under the hypotheses of Theorem 1, the degree of approximation of f, by Borel's exponential means of its Fourier series, is given by*

(1.12) $$\sup_{0 \leqslant x \leqslant 2\pi} | f(x) - T(p, x) | = O(p^{-\alpha}) .$$

## 2. LEMMA

We shall use the following lemma in the proof of Theorem 1 :

**LEMMA.** *Let* $0 < t \leqslant \pi$ *and* $q > 0$ . *Then*

$$(1 + q)^{-n} (1 + q^2 + 2q \cos t)^{n/2} < \exp(-Ant^2) .$$

For a proof, see Chandra ( [3]; Lemma 1 ) .

## 3. PROOF OF THEOREM 1

Proceeding as in [3], we obtain

$$\pi ( t(n,q,x) - f(x) )$$

$$= \int_0^\pi \frac{\phi_x(t)}{\sin \frac{1}{2} t} \left\{ (1+q)^{-n} \sum_{k=0}^{n} \binom{c}{k} q^{n-k} \sin (k + \tfrac{1}{2}) t \right\} dt .$$

Now we express the integral $\int_0^\pi$ as a sum of sub-integrals $\int_0^{1/n}$ , $\int_{1/n}^{\delta}$ and $\int_\delta^\pi$ , $0 < \delta < \pi/4$, and denote these by $I_1$ , $I_2$, and $I_3$ , respectively . Then

$$\sup_{0 \leqslant x \leqslant 2\pi} | I_1 |$$

$$\leqslant \sup_{0 \leqslant x \leqslant 2\pi} \int_0^{1/n} \frac{| \phi_x (t) |}{\sin \frac{1}{2}t} \left\{ (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \right\} dt$$

$$= \sup_{0 \leqslant x \leqslant 2\pi} \int_0^{1/n} | \phi_x (t) / \sin \tfrac{1}{2}t | \, dt$$

$$= O \left\{ \int_0^{1/n} t^{\alpha-1} \, dt \right\} \quad (\text{by } (1.1))$$

$$= O ( n^{-\alpha} )$$

and, proceeding as in Chandra [ 1, ( 3.2 ) ],

$$\sup_{0 \leqslant x \leqslant 2\pi} | I_3 | \quad \sup_{0 \leqslant x \leqslant 2\pi} \int_\delta^{\pi} \frac{| \phi_x (t) |}{\sin \frac{1}{2}t} P ( q, t, n )$$

$$| \sin \{ \tfrac{1}{2} t + nQ ( q, t ) \} | \, dt$$

$$= O \left\{ \int_\delta^{\pi} t^{\alpha-1} \exp ( - Ant^2 ) \, dt \right\}$$

( by the lemma )

$$= O ( n^{-1} ) \int_\delta^{\pi} t^{\alpha-2} \left\{ - \frac{\partial}{\partial t} \exp ( - Ant^2 ) \right\} dt$$

$$= O ( n^{-1} ) .$$

Next

$$| I_2 | \leqslant 2 \left| \int_{1/n}^{\delta} t^{-1} \phi_x (t) \, P ( q, t, n ) \sin \{ nQ ( q, t ) + \tfrac{1}{2} t \} \, dt \right|$$

$$+ \left| \int_{1/n}^{\delta} \phi_x (t) (\operatorname{cosec} \tfrac{1}{2}t - 2/t ) P ( q, t, n ) \sin \{ \tfrac{1}{2}t + nQ ( q, t ) \} \, dt \right|$$

$$= 2\,|\,I_{2,1}\,| + |\,I_{2,2}\,|\,, \text{ say },$$

and

$$\sup_{0 \leqslant x \leqslant 2\pi} |\,I_{2,2}\,| = O\left\{ \int_{1/n}^{\delta} t^{1+\alpha} \quad P(q, t, n)\; dt \right\}$$

$$= O(n^{-1}) \int_{1/n}^{\delta} t^{\alpha} \left\{ -\frac{\partial}{\partial t} \; exp\,(-Ant^2) \right\} dt \quad (\text{ as for } I_3)$$

$$= O(n^{-1})\;.$$

Since

(3.1) $\quad \phi_x(t) + M\,t^{\alpha} \geqslant 0\,,$

$$|\,I_{2,1}\,| \leqslant \left| \int_{1/n}^{\delta} t^{-1}.(\,\phi_x\,(t) + Mt^{\alpha}\,)\,P(\,q,t,n,\,)\;sin\;\{nQ\,(q,t) + \tfrac{1}{2}t\}\;dt \right| +$$

$$+ M \left| \int_{1/n}^{\delta} t^{\alpha-1}\,P\,(\,q,\,t,\,n\,)\;\;sin\;\;\{\,nQ\,(\,q,\,t\,) + \tfrac{1}{2}t\,\}\;dt \right|$$

$$= |\,I_{2,1,1}\,| + M\,|\,I_{2,1,2}\,|\;,\; \text{say}\;.$$

$P(q, t, n)$ is positive and non–increasing in $(1/n, \delta)$ ; therefore, by the second mean value theorem,

$$I_{2,1,2} = u^{1-\alpha}\,P(\,q,n^{-1},n\,) \int_{1/n}^{\delta'} sin\,(\,nQ\,(\,q,t\,) + \tfrac{1}{2}t\,)\;dt \quad (1/n < \delta' < \delta)\,.$$

Since

$$\left\{ \frac{d}{dt}\;Q\,(\,q,\,t\,) + \frac{1}{2n} \right\}^{-1} = \left\{ \frac{q\;cos\;t + cos\;2t}{1 + q^2 + 2q\;cos\;t} + \frac{1}{2n} \right\}^{-1}$$

is an increasing function of $t$ ,

$$\int_{1/n}^{\delta'} sin\;\{\,nQ\,(\,q,\,t\,) + \tfrac{1}{2}\,t\,\}\;\;dt$$

$$= \int_{1/n}^{\delta'} \left\{ n \frac{d}{dt} Q(q,t) + \tfrac{1}{2} \right\}^{-1} \frac{d}{dt} \left\{ -\cos(nQ(q,t) + \tfrac{1}{2}t) \right\} dt$$

$$= n^{-1} \int_{1/n}^{\delta'} \left\{ \frac{d}{dt} Q(q,t) + \frac{1}{2n} \right\}^{-1} \frac{d}{dt} \left\{ -\cos(nQ(q,t) + \tfrac{1}{2}t) \right\} dt$$

$$(1/n < \eta < \delta')$$

$$= O(n^{-1}),$$

we obtain

$$\sup_{0 \leqslant x \leqslant 2\pi} | I_{2,1,2} | = O(n^{-\alpha})$$

Finally, in view of (3.1) and (1.10), and the second mean value theorem, we have

$$I_{2,1,1} = n \{ \phi_x(1/n) + M n^{-\alpha} \} P(q, n^{-1}, n)$$

$$\int_{1/n}^{\delta''} \sin \{ nQ(q,t) + \tfrac{1}{2}t \} dt \quad (1/n < \delta'' < \delta)$$

$$= O(n^{-\alpha}).$$

Thus

$$\sup_{0 \leqslant x \leqslant 2\pi} | I_{2,1,1} | = G(n^{-\alpha}).$$

Collecting the results, we obtain a proof of Theorem 1.

## 4. PROOF OF THEOREM 2

Proceeding as in [2], we have

$$\pi(T(p,x) - f(x)) = \int_0^{\pi} \frac{\phi_x(t) \sin \{ \frac{1}{2}t + p \sin t \}}{\sin \frac{1}{2}t \exp(p(1 - \cos t))} dt$$

Now we express the integral $\int_0^{\pi}$ as a sum of sub-integrals $\int_0^{1/p}$,

$\int_{1/p}^{\delta}$ and $\int_{\delta}^{\pi}$ , $0 < \delta < \pi/4$ , and denote these by $J_1$, $J_2$ and $J_3$ , respectively . Then proeeeding as in [2] . we obtain that

$$\sup \{ \mid J_1(x) \mid : 0 \leqslant x \leqslant 2\pi \} = O(p^{-\alpha})$$

and

$$\sup \{ \mid J_3(x) \mid : 0 \leqslant x \leqslant 2\pi \}$$

$$\leqslant \sup_{0 \leqslant x \leqslant 2\pi} \int_{\delta}^{\pi} \frac{\mid \phi_x(t) \mid}{2p \sin t \sin \frac{1}{2}t} \left\{ - \frac{\partial}{\partial t} \left\{ (-2p \sin^2 \tfrac{1}{2}t) \right\} dt \right.$$

$$= O(p^{-1}) \int_{\delta}^{\pi} t^{\alpha-2} \left\{ - \frac{\partial}{\partial t} \exp(-2p \sin^2 \tfrac{1}{2}t) \right\} dt$$

$$= O(p^{-1}) \text{ .}$$

And

$$\mid J_2 \mid \leqslant 2 \left| \int_{1/p}^{\delta} t^{-1} \phi_x(t) \frac{\sin(\frac{1}{2}t + p \sin t)}{\exp(p(1 - \cos t))} dt \right|$$

$$+ \left| \int_{1(p}^{\delta} \phi_x(t) (\operatorname{cosec} \tfrac{1}{2}t - 2/t) \frac{\sin(\frac{1}{2}t + p \sin t)}{\exp(p(1 - \cos t))} dt \right|$$

$$= 2 \mid J_{2,1} \mid + \mid J_{2,2} \mid \text{, say ,}$$

where

$$\sup_{0 \leqslant x \leqslant 2\pi} \mid J_{2,2} \mid = O \left\{ \int_{1/p}^{\delta} t^{1+\alpha} \exp(-2p \sin^2 \tfrac{1}{2}t) \, dt \right\}$$

$$= O(p^{-1}) \int_{1/p}^{\delta} t^{\alpha} \left\{ - \frac{\partial}{\partial t} \exp(-2p \sin^2 \tfrac{1}{2}t) \; dt \right\}$$

$$= O(p^{-1})$$

and, using the arguments used for estimating

$$\sup_{0 \leqslant x \leqslant 2\pi} | I_{2,1} |$$

of Theorem 1, we easily obtain

$$\sup_{0 \leqslant x \leqslant 2\pi} | J_{2,1} | = O ( p^{-\alpha} ) .$$

This proves Theorem 2 completely .

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