Jñānābha, Vol. 17, 1987

ON $|C, \alpha, \gamma|_k$ SUMMABILITY FACTORS OF FOURIER SERIES

By

Ashok Ganguly

Department of Mathematics,

G. S. Institute of Technology and Science,

Indore–452 003, M. P., India

(Received : May 28, 1985 ; Revised : July 3, 1986)

1. INTRODUCTION

Let Σu_n be a given series with the sequence of partial sums $\{S_n\}$ and let $t_n = t_n^0 = nu_n$. By $\{S_n^\alpha\}$ and $\{t_n^\alpha\}$ we denote the n-th Cesãro means of order α ($\alpha > -1$) of the sequences $\{S_n\}$ and $\{t_n\}$ respeetively. Thus

$$S_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} s_\nu ,$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!}.$$

The series Σu_n is said to be summable $|C, \alpha, \gamma|_k$, $k \geqslant 1$, $\alpha > -1$ and $\gamma \geqslant 0$, if

$$(1.1) \quad \sum_n n^{k\gamma + k - 1} \left| S_n^\alpha - S_{n-1}^\alpha \right|^k < \infty \quad \text{(Flett [5])}.$$

By virtue of the identity

$$t_n^\alpha = n \left(S_n^\alpha - S_{n-1}^\alpha \right) \quad \text{(Kogbetliantz [7])}$$

(1.1) can also be written as

(1.2) $\sum_n n^{k\gamma-1} \mid t_n^\alpha \mid^k < \infty$.

For $\gamma = 0$, the summability $| C, \alpha, \gamma |_k$ reduces to the summability $| C, \alpha |_k$ (see [6]) ; for $\gamma = 0$ and $k = 1$, it reduces to the summability $| C, \alpha |$.

2. THE MAIN RESULT

Let the Fourier series associated with $f(t)$ be

$$1/2\, a_0 + \sum_{n=1}^{\infty} (a_n \, Cos\, nt + b_n \, Sin\, nt) = \sum_{n=0}^{\infty} A_n (t) .$$

We write

$$\phi (t) = 1/2 \, \{ f (x + t) + f (x - t) - 2S \} .$$

A sequence $\{ \mu_n \}$ is said to be convex, if $\Delta^2 \mu_n \geqslant 0$; $n = 1, 2, \ldots$, where $\Delta \mu_n = \mu_n - \mu_{n+1}$ and $\Delta^2 \mu_n = \Delta (\Delta \mu_n)$.

The objet of this paper is to prove the following :

Theorem . *If* $\{ \mu_n \}$ *is a convex sequence such that*

$\sum \frac{\mu_n}{n} < \infty$, *then the series* $\sum \frac{\mu_n A_n (t)}{n^{1-\alpha+\gamma} (Log\, n)^{\beta/k}}$ *at* $t = x$

is summable $| C, \alpha, \gamma |_k$, *where* $0 \leqslant \gamma < \alpha < 1$ *and* $k \geqslant 1$, *provided that*

$$(2.1) \quad \int_0^t | \phi(u) |^k du = O \left\{ t \left(\log \frac{1}{t} \right)^\beta \right\}, \ \beta \geqslant 0 , \ t \to 0 .$$

For $k = 1, \gamma = 0$, our theorem reduces to the result of Dikshit [4] which is an extension of a theorem of Cheng [2] .

3. PREREQUISITES

We require the following lemmas :

Lemma 1 [3] . *If* $0 < \alpha < 1, \ 0 < t < 2\pi$, *and*

$$R_n^\alpha (t) = \sum_{r=1}^{n} A_{n-r}^{\alpha-1} \, r \cos rt \ ,$$

then

$$R_n^\alpha (t) = \begin{cases} O (n^2) \text{ for all } t > 0 , \\ O (nt^{-\alpha}) \text{ for } t > 1/n . \end{cases}$$

Lemma 2 [1] . *If* $0 \leqslant \alpha \leqslant 1$ *and* $0 \leqslant m \leqslant n$, *then*

$$\left| \sum_{\nu=0}^{m} A_{n-\nu}^{\alpha-1} a_\nu \right| \leqslant \max_{0 \leqslant r \leqslant m} \left| \sum_{\nu=0}^{r} A_{r-\nu}^{\alpha-1} a_\nu \right| .$$

Lemma 3 . *Let* $0 < \alpha < 1$ *and* $0 < t \leqslant 2\pi$, *and write*

$$M_a^\alpha (t) = \frac{1}{A_n^\alpha} \sum_{\nu=2}^{n} A_{n-\nu}^{\alpha-1} \frac{\mu_\nu}{(\log \nu)^{\beta/k}} \nu^{\alpha-\gamma} \operatorname{Cos} \nu t , \ \beta \geqslant 0 ,$$

$0 \leqslant \gamma < \alpha < 1$ *and* $k \geqslant 1$. *Then*

$$M_n(t) = \begin{cases} O\left\{ n^{-\alpha} \sum_{\nu=2}^{n} \frac{\nu^{1+\alpha-\gamma} \Delta\mu_\nu}{(\log \nu)^{\beta/k}} \right\} + O\left\{ \frac{n^{1-\gamma}\ \mu_n}{(\log n)^{\beta/k}} \right\} & \text{for } 0 < t \leqslant 1/n, \quad (3.1) \\ O\left\{ (nt)^{-\alpha} \sum_{\nu=2}^{n} \frac{\nu^{\alpha-\gamma}}{(\log \nu)^{\beta/k}} \Delta\mu_\nu \right\} + O\left\{ \frac{\mu_n}{n^{\gamma}\ t^{\alpha}\ (\log n)^{\beta/k}} \right\} & \text{for } t > 1/n. \quad (3.2) \end{cases}$$

By applying Abel's transformation and taking the help of lemmas 1 and 2, Lemma 3 can easily be proved.

Lemma 4. *If the condition (2.1) holds, then (for $k \geqslant 1$)*

(i) $$\left\{ \int_0^{1/n} |\phi(t)|\, dt \right\}^k = O\left\{ \frac{(\log n)^\beta}{n^k} \right\} ;$$

and, for $k \geqslant 1$ and $0 < \alpha < 1$,

(ii) $$\left\{ \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^\alpha} dt \right\}^k = O\left\{ (\log n)^\beta \right\} .$$

These results follow by an integration by parts or by an application of Hölder's inequality, as necessary.

Lemma 5. *If $\{\mu_n\}$ is a convex sequence such that $\Sigma \frac{\mu_n}{n} < \infty$, then*

(i) $$\sum_{n=1}^{m} \log(n+1)\, \Delta\mu_n = O(1) \text{ as } m \to \infty, \text{ (See Pati [8]) ;}$$

further, the hypotheses on $\{\mu_n\}$ imply

(ii) $$\sum_{n=1}^{m} \Delta\mu_n = O(1),\ m \to \infty,$$

(iii) $\sum_{n=1}^{m} \frac{(\mu_n)^k}{n} = O(1), \; m \to \infty .$

The proof of (ii) and (iii) follow easily .

4. PROOF OF THE THEOREM

We denote the n-th Cesàro mean of order α of the sequence $\{ n^{\alpha-\gamma} \mu_n A_n(x) (\log n)^{-\beta/k} \}$ by $C_n^{\alpha}(x)$. Then we have to show that

$$\Sigma\, n^{k\gamma-1} \mid C_n^{\alpha}(x) \mid^k < \infty \quad \text{for } k \geqslant 1 .$$

We have

$$C_n^{\alpha}(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \frac{1}{A_n^{\alpha}} \sum_{\nu=2}^{n} \frac{A_{n-\nu}^{\alpha-1} \nu^{\alpha-\gamma} \mu_\nu \cos \nu t}{(\log \nu)^{\beta/k}} dt$$

$$= \frac{2}{\pi} \left(\int_0^{1/n} + \int_{1/n}^{\pi} \right) \phi(t)\, M_n^{\alpha}(t)\, dt$$

$$= Q_n^1 + Q_n^2 , \quad say .$$

In view of Minkowski's inequality , it is sufficient to prove that

(4.1) $\Sigma\, n^{k\gamma-1} \mid Q_n^1 \mid^k < \infty$, and

(4.2) $\Sigma\, n^{k\gamma-1} \mid Q_n^2 \mid^k < \infty$.

Proof of (4.1) . Using (3.1) we have

$$\sum_{n=2}^{m} n^{k\gamma-1} \left| Q_n^1 \right|^k \leqslant \sum_{n=2}^{m} n^{k\gamma-1} \left[\frac{2}{\pi} \int_0^{1/n} \left| \phi(t) \right| dt \right.$$

$$\left. \left\{ O\left(n^{-\alpha} \sum_{\nu=2}^{n} \frac{\nu^{1+\alpha-\gamma} \Delta \mu_\nu}{(\log \nu)^{\beta/k}} \right) + O\left(\frac{n^{1-\gamma} \mu_n}{(\log n)^{\beta/k}} \right) \right\} \right]^k$$

$$= \left\{ O\left[\sum_{n=2}^{m} n^{k\gamma-1} \left(\int_0^{1/n} \frac{|\phi(t)|}{n^{\alpha}} \sum_{\nu=2}^{n} \frac{\nu^{1+\alpha-\gamma} \Delta \mu_\nu}{(\log \nu)^{\beta/k}} dt \right)^k \right]^{1/k} \right.$$

$$\left. + O\left[\sum_{n=2}^{m} n^{k\gamma-1} \left(\int_0^{1/n} \frac{|\phi(t)| \, n^{1-\gamma} \mu_n}{(\log n)^{\beta/k}} dt \right)^k \right]^{1/k} \right\}^k$$

$$= \left\{ M_1^{1/k} + M_1^{1/k} \right\}^k, \quad say .$$

Now, applying Lemmas 4 (i) and 5 (i), we get

$$M_1 = O\left[\sum_{n=2}^{m} \frac{1}{n^{1+k(\alpha-\gamma)}} \left(\int_0^{1/n} |\phi(t)| \, dt \right)^k \left(\sum_{\nu=2}^{n} \frac{\nu^{1+\alpha-\gamma} \Delta \mu_\nu}{(\log \nu)^{\beta/k}} \right)^k \right]$$

$$= O\left[\sum_{n=2}^{m} \frac{1}{n^{1+k(\alpha-\gamma)}} \cdot \frac{(\log n)^\beta}{n^k} \left(\sum_{\nu=2}^{n} \frac{\nu^{k(1+\alpha-\gamma)} \Delta \mu_\nu}{(\log \nu)^\beta} \right) \left(\sum_{\nu=2}^{n} \Delta \mu_\nu \right)^{k-1} \right]$$

$$= O\left[\sum_{\nu=2}^{m} \frac{\nu^{k(1+\alpha-\gamma)} \Delta \mu_\nu}{(\log \nu)^\beta} \cdot \sum_{n=\nu}^{m} \frac{(\log n)^\beta}{n^{1+k(1+\alpha-\gamma)}} \right]$$

$$= O\left[\sum_{\nu=2}^{m} \Delta \mu_\nu \right]$$

$= O$ (1) *as* $m \to \infty$, by Lemma 5 (ii) .

Next , using Lemma 4 (i) again, we get

$$M_2 = O\left[\sum_{n=2}^{m} \frac{n^{k-1}\ \mu_n^k}{(\log n)^\beta}\left(\int_0^{1/n} |\phi(t)|\, dt\right)^k\right]$$

$$= O\left[\sum_{n=2}^{m} \frac{\mu_n^k}{n}\right]$$

$= O$ (1) as $m \to \infty$, by Lemma 5 (iii).

Proof of (4.2) : Applying (3.2), we obtain

$$\sum_{n=2}^{m} n^{k\gamma-1} |Q_n^2|^k \leqslant \sum_{n=2}^{m} n^{k\gamma-1}\left[\frac{2}{\pi}\int_{1/n}^{\pi} |\phi(t)|\, dt\right.$$

$$\left.\left\{O\left((nt)^{-\alpha}\sum_{\nu=2}^{n}\frac{\nu^{\alpha-\gamma}\,\Delta\mu_\nu}{(\log \nu)^{\beta/k}}\right)+O\left(\frac{\mu_n}{t^\alpha\, n^\gamma\,(\log n)^{\beta/k}}\right)\right\}\right]^k$$

$$= \left\{O\left[\sum_{n=2}^{m} n^{k\gamma-1}\left(\frac{1}{n^\alpha}\int_{1/n}^{\pi}\frac{|\phi(t)|}{t^\alpha}\sum_{\nu=2}^{n}\frac{\nu^{\alpha-\gamma}\,\Delta\mu_\nu}{(\log \nu)^{\beta/k}}dt\right)^k\right]^{1/k}\right.$$

$$\left.+O\left[\sum_{n=2}^{m} n^{k\gamma-1}\left(\frac{1}{n^\gamma}\int_{1/n}^{\pi}\frac{|\phi(t)|}{t^\alpha}\,\frac{\mu_n}{(\log n)^{\beta/k}}\,dt\right)^k\right]^{1/k}\right\}^k$$

$$= \left\{N_1^{1/k} + N_2^{1/k}\right\}, \text{ say}.$$

Using Lemmas 4 (ii), and 5 (i), we get

$$N_1 = O\left[\sum_{n=2}^{m}\frac{1}{n^{1+k(\alpha-\gamma)}}\left(\int_{1/n}^{\pi}\frac{|\phi(t)|}{t^\alpha}\,dt\right)^k\left(\sum_{\nu=2}^{n}\frac{\nu^{\alpha-\gamma}\,\Delta\mu_\nu}{(\log \nu)^{\beta/k}}\right)^k\right]$$

$$= O\left[\sum_{n=2}^{m}\frac{(\log n)^\beta}{n^{1+k(\alpha-\gamma)}}\left(\sum_{\nu=2}^{n}\frac{\nu^{k(\alpha-\gamma)}\,\Delta\mu_\nu}{(\log \nu)^\beta}\right)\left(\sum_{\nu=2}^{n}\Delta\mu_\nu\right)^{k-1}\right]$$

$$= O\left[\sum_{\nu=2}^{m} \frac{\nu^{k(\alpha-\gamma)}\,\Delta\mu_\nu}{(\log \nu)^\beta} \sum_{n=\nu}^{m} \frac{(\log n)^\beta}{n^{1+k(\alpha-\gamma)}}\right]$$

$$= O\left[\sum_{\nu=2}^{m} \Delta\mu_\nu\right]$$

$= O(1)$, as $m \to \infty$, by Lemma 5 (ii) .

Further, applying Lemma 4 (ii) again, we have

$$N_2 = O\left[\sum_{n=2}^{m} \frac{\mu_n^k}{n} \frac{1}{(\log n)^\beta} \left(\int_{1/n}^{\pi} \frac{|\phi(t)|}{t^\alpha}\,dt\right)^k\right]$$

$$= O\left[\sum_{n=2}^{m} \frac{\mu_n^k}{n}\right]$$

$= O(1)$, as $m \to \infty$, by Lemma 5 (iii) .

Thus the proof of the theorem is complete .

Acknowledgements

The author is extremely grateful to Professor H. M. Srivastava and the referee for suggesting a number of improvemnnts in the paper and for correcting several errors in the revised version .

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