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A DEGREE THEORY FOR WEAKLY CONTINUOUS MULTIVALUED MAPS IN REFLEXIVE BANACH SPACES*

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ABSTTRACT

In this paper we propose a degree theory for multivalued weakly upper semicontinuous maps with convex values in a reflexive Banach space with Schauder basis.

INTRODUCTION

A degree theory for singlevalued weakly continuous maps was developed in [4]. The aim of this note is to introduce a degree theory for multivalued weakly upper semicontinuous maps (see Definition 1. 5). Our results extend the degree theory of singlevalued weakly continuous maps of [4].

The paper is divided into three parts.

In the first part we give the notations and the basic definitions to be used in the sequel and we introduce the so-called condition (A) which plays the same role in our theory as the usual condition "the

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map has no fixed points on the boundary" in the case of maps upper semicontinuous with convex compact values.

The second part is devoted to the degree theory for the multivalued weakly upper semicontinuous maps with convex values, satisfying the condition (A) with respect to some Schauder basis in reflexive Banach spaces.

Finally, in the third part we introduce a class of multivalued maps which satisfies the condition (A) with respect to every orthonormal basis of a Hilbert space.

1. NOTATIONS, DEFINITIONS AND PRELIMINARY RESULTS

We recall that a multivalued map of a set X into a set Y is a triple (G, X, Y), where G, the graph of T, is a subset of $X \times Y$ such that $Tx = \{y \in Y; (x, y) \in G\}$ is nonempty for each $x \in X$. $TX = \{Tx : x \in X\}$ is the range of T, while X is its domain. We shall use the symbol $T : X \multimap Y$ to indicate a multivalued map. If $D \subset X$ then $TD = \{Tx : x \in D\}$.

Let X and Y be topological spaces and $T : X \to Y$.

Definition 1. 1.

The map T is called upper semicontinuous (u. s c.) at $x_0 \in X$ if for any open set V containing Tx_0 there exists a neighbourhood U of x_0 such that $x \in U$ implies $Tx \subset V$.

Definition 1. 2.

The map T is called u. s. c. on X if it is u. s. c. at each point $x \in X$ and Tx is compact for every $x \in X$.

Definition 1. 3.

The map T is called closed if its graph is closed in $X x Y$.

Definition 1. 4.

The map T is called closed if its graph is closed in $X \times Y$.

A fixed point of a multivalued map $T : X \to Y$ is a point $x \in X$ such that $x \in Tx$.

Let X be a real refiexive Banach space with norm $\| \cdot \|$, which admits a Schauder basis. By B_r we denote the ball $\{x \in X : \| x \| \leqslant r\}$, where r is a real positive number, with ∂B_r the sphere $\{x \in X: \| x \| = r\}$. We shall also use the symbols "—o" and "$\rightharpoonup$" to denote the strong and the weak convergence in X respectively. Moreover, we shall denote by τ the weak topology on X.

Definition 1. 5.

The map $T : B_r \to X$ is called weakly upper semicontinuons (*w. u. s. c.*) *at $x_0 \in S_r$, if it is u. s. c. at x_0, with respect to the weak topology τ.*

Definition 1. 6.

The map $T : S_r \to X$ is called w. u. s. c. on B_r, if it is w. u. s. c. at each point $x \in B_r$ and Tx is weakly compact (i. e. compact with respect to τ) *for every $x \in S_r$.*

Definition 1. 7.

The map $T : B_r \to X$ is called weakly closed, if its graph is closed in $B_r \times X$ with respect to the weak topology on the product.

Remark 1. 1.

If $T : S_r \to X$ is weakly closed then (see [1]) :

$$\left.\begin{array}{r} \{x_m\} \subset S_r, \ x_n \to x \\ y_n \to y \\ \forall\ n \in N, y_n \in Tx_n \end{array}\right\} \Rightarrow y \in Tx.$$

Definition 1. 8.

Let $D \subset X$. The map $T: D \to X$ is called bounded on D, if it maps bounded sets of D into bounded sets of X.

Remark 1. 2.

Let $D \subset X$. If D is weakly compact then D is bounded and weakly closed (*i. e.* closed with respect to τ), by the reflexivity of X.

Remark 1. 3.

If $T : S_r \to X$ is *w. u. c.* on S_r then it is weakly closed (see [1]).

Remark 1. 4.

If $T : X \to X$ is *w. u. s. c.* then it sends weakly compact sets of X into weakly compact sets of X (see [1]).

Remark 1. 5.

If $T : B_r \to X$ is *w. u. s. c.* on S_r then Tx is closed for any $x \in B_r$.

Proposition 1. 1.

Let $D \subset X$. If $T : D \to X$ is w. u. s. c. and D is weakly closed then T is bounded on D.

Proof : Given a bounded subset $K \subset D$, its weak closure $\overline{\overline{K}}$ (*i. e.* the closure of K with respect to τ) is weakly compact, by the reflexivity of X. Moreover, $\overline{\overline{K}}$ is contained in D, since D is weakly closed. Consequently, $T\overline{\overline{K}}$ is weakly compact (see Remark 1. 4). Form the Remark 1. 2 it follows that $T\overline{\overline{K}}$ is bounded and weakly closed. Hence $TK \subset T\overline{\overline{K}}$ is bounded "

Let X be a real reflexive Banach space, $\{e_i\}$ a Schauder basis of X and $\{\phi_i\}$ the corresponding biorthogonal system in the dual space X^*, *i. e.* $\phi_i(e_j) = \delta_{ij}$. Given $n \in N$, by $\partial_m S_r$ we denote the

n-dimensional sphere $\{ x \in \partial S_r : \sum_{i=1}^{n} | \phi_i (x) |^2 = r^2 \}$.

We say that the map $T : B \to X$ satisfies the condition (A), with respect to the Schauder basis $\{e_i\}$, if there exists $n_0 \in N$ such that, for all $n \geqslant n_0$ and for all $x \in \partial_n S_r$ and $y \in Tx$, there exists $j \leqslant n$ such that $\phi_j (y) \neq \phi_j (x)$.

In particular, we say that the map $T ; B \to X$ satisfies the condition (B), if it satisfies the condition (A) with $n_0 = 1$.

Finally, the map $T : B \to X$ satisfies the condition (C), if $x \notin Tx$ for all $x \in \partial S_r$.

The following examples prove the mutual independence of the condition (A) and (C).

Example 1. 1. ((A) does not imply (C)).

Let H be a real and separable Hilbert space with inner product $(,)$ and orthonormal basis $\{e_i\}$. Fix $a \in H$ such that $\| a \| = 2r - \varepsilon$, where $\varepsilon < r$ is a real and positive number. We define $Tx = S_\varepsilon + a - x$, where $S_\varepsilon = \{ z \in H : \| z \| \leqslant \varepsilon\}$, for all $x \in S_r$. The map T has a fixed point on ∂S_r. Let $x = [r/(2r - \varepsilon)]a$ and $z = [\varepsilon/(2r - \varepsilon)]a$, then we have $\bar{x} = z + a - \bar{x}$ with $\| x \| = r$ and $\| z \| = \varepsilon$. Consequently, $\bar{x}$ is a fixed point of T on ∂S_r and thus (C) does not hold. On the other hand the map T verifies the condition (A) with respect to every orthonormal basis $\{e_i\}$ with the property that $(a, e_j) \neq 0$ for infinitely many indices j. Suppose now that T does not satisfy the condition (A). Then one can find x and z such that $(z + a - x) \in Tx$, $x \in \partial_n S_r$, and $(z + a - x, e_j) = (x, e_j)$, for $j = 1,\dots,n$. By the second and, third property, $\sum_{j=1}^{n} [(z, e_j) + (a, e_j)]^2 = 4r^2$; on the other hand

$\sum_{j=1}^{n} [(z, e_j) + (a, e_j)]^2 < 4r^2$, since $(a, e_j) \neq 0$ for infinitely many j.

This contradiction completes the **proof.**

Example 1. 2. ((C) does not imply (A))

Let $\{e_i\}$ be an orthonormal basis of H and $\varepsilon < r$ a real positive number. We define, for all $x \in S_r$

$$Lx = \{h \in H : h = x + \sum_{i=1}^{\infty} (y, e_i)\, e_{i+1} \text{ for some } y \in S_\varepsilon + x\}.$$

It is easy to verify that the map L is without fixed points on ∂S_r. The map L, however, does not satisfy the condition (A) with respect to the orthonormal basis $\{e_i\}$. In fact, for all $n \in N$, the element $re_n \in \partial_n S_r$ is such that

$$(re_n, e_j) = (re_n + \sum_{i=1}^{\infty} (re_n, e_i)\, e_{i+1}, e_j) \text{ for } j = 1, \quad , n.$$

Remark 1. 6.

Obviously, in the definition of condition (A), the choice of the index n_0 depends, in general, on the basis. Moreover the example 1. 1 shows that n_0 may tend to $+\infty$, if the orthonormal basis $\{e_i\}$ changes.

Proposition 1 2.

If $T : B_r \to X$ is w. u. s. c. and does not satisfy the condition (A) with respect to a Schauder basis $\{e_i\}$, then T has fixed point in S_r.

Proof : If T does not satisfy the condition (A) there exists an infinite sequence $\{n_k\}$ of positive intergers with $n_k \to +\infty$ and a sequence $\{x^k\}$ with $x^k \in \partial_{n_k} B_r$, for all $k \varepsilon N$, such that $\phi_j(y) = \phi_j(x^k)$ for $j=1,..,n_k$ for some $y \varepsilon Tx_k$. By contruction, there exists $ze^{n_k} \in Tx^{n_k}$,

such that $z^{n_k} = x^{n_k} + y^{n_k}$ with $x^{n_k} = \sum_{i=1}^{n_k} \phi_i (x^{n_k}) e_i$

and $y^{n_k} = \sum_{i=n_k+1}^{\infty} \phi_i (z^{n_k}) e_i$, since $\{x^{n_k}\}$ is bounded there exists a subsequence $\{x^{n_{k(s)}}\}$ of $\{x^{n_k}\}$ such that $x^{n_{k(s)}} \rightharpoonup x \in S_r$. Moreover, the corresponding subsequence $\{z^{n_{k(s)}}\}$ of $\{z^{n_k}\}$ converges also weakly to x. In fact, this follows from the bounded ness of the sequence $\{z^{n_{k(s)}}\}$ and the map T, (see Proposition 1.1), since $(z^{n_{k(s)}}, e_j) \to (x, e_j)$ for all $j \in N$. Finally, since the graph of T is weakly closed, one has $x \in Tx$ (see Remark 1 1).

2. Definition of a Topological Degree for Multivalued W. U. S. C. Maps

Let $T : B_r \to X$ be w. u. s. c., and $\{e_i\}$ a Schauder basis in X with biorthogonal system $\{\phi_i\}$ (see before). Let $X_n = [e_1, \ , e_n]$ the linear hull of the first n basis vectors, B_r^n the sphere of radius r in X_n, and ∂B_r^n its boundary For fixed $n \in N$, we define $T_n : B_r^n \to X_n$ by

$$T_n (x_1, \ldots, x_n) = \{ (y_1, \ , y_n) : y \in Tx \},$$

where $x_i = \phi_i(x)$ and $y_i = \phi_i(y)$ for $i = 1, \ldots, n$.

If $T : S_r \to X$ is w. u. s. c. and satisfies the condition (A) with respect to the Schauder basis $\{e_i\}$, then $T_n : B_r^n \to X_n$ is fixed point free on

∂B_r^n, for all $n \geqslant n_0$. In fact, if we had $x \in T_n x$ for some $x \in \partial B_r^n$ there would exist $y \in Tx$ with $x = \sum_{i=1}^{n} \phi_i(x) e_i$ and hence $\phi_j(y) = \phi_j(x)$ for $j = 1, \ldots, n$, contradicting the condition (*A*). Moreover, T_n is u. s. c. for all $n \in N$, as composition of map T with imbeddings and projections.

Definition 2. 1.

Let $T : B_r \to X$ be w. u s. c, with convex values. If T satisfies the condition (*A*), with respect to the Schauder basis $\{e_i\}$, we define Deg($I-T, S_r$, 0), the degree of T on S_r with respect to 0, as follows: Let $\hat{Z}$ be the set of all integers togethers with $\{+\infty\}$ and $\{-\infty\}$. Then Deg ($I-T, S_r$, 0) is defined to be the subset of $\hat{Z}$ given by:

Deg ($I-T, S_r$, 0) $= \{\gamma \in \hat{Z}$/there exists an infinte sequence $\{n_k\}$

of positive integers with $n_k \to \infty$ such that Deg $(I-T_{n_k}, B_r^{n_k}, 0) \to \gamma\}$.

Remark 2. 1.

Deg ($I-T, S_r$, 0) $\neq \phi$, since $\hat{Z}$ is compact. In particular, if γ is a (finite) integer, then Deg $(I-T_{n_k}, B_r^{n_k}, 0) \to \gamma$ iff Deg $(I-T_{n_k}, B_r^{n_k}, 0) = \gamma$ for all but a finite number of n_k.

Remark 2. 2.

The degree Deg $(I-T_{n_k}, B_r^{n_k}, 0)$ used in Definition 2. 1 is the Cellina -Lasota degree [3]. It is well defined for all $n_k \geqslant n_0$.

In fact, the map T is w. u. s. c., bounded, with convex and closed values (see Proposition 1. 1 and Remark 1. 5). Hence

$I\text{-}T_{n_k} : B_r^{n_k} \rightarrow X_{n_k}$ is u. s. c., with convex and compact values.

The following theorems gather the main properties of the degree of Definition 2. 1.

Theorem 2. 1. (Solution property).

Let $T : B_r \rightarrow X$ be w. u. s. c., with convex values, and such that condition (A) holds with respect to the Schauder basis $\{e_i\}$. If Deg $(I\text{-}T, B_r, 0) \neq \{0\}$, then there exists an element $x \in B_r$ such that $x \in Tx$.

Proof : If Deg $(I\text{-}T, B_r, 0) \neq \{0\}$ there exists an infinite sequence $\{n_k\}$ of positive integers with $n_k \rightarrow +\infty$ such that deg $(I\text{-}T_{n_k}, B_r^{n_k}, 0) \neq 0$. By the solution property of the Cellina-Lasota degree it follows that for every n_k there exists $x^{n_k} \in B_r^{n_k}$ such that $x^{n_k} \in T_{n_k} x^{n_k}$. The remaining part of the proof is analogues to that of the Proposition 1. 2.

Theorem 2. 2. (Homotopy invariance).

Let $H : B_r \times [0\ 1] \rightarrow X$ be a w. u. s. c. map such that $H(x, t)$ ts convex for every $(x, t) \in B_r \times [0, 1]$. If the family of the maps $H_t = H(\cdot\,, t) : B_r \rightarrow X$ verifies the condition (A) for all $t \in [0, 1]$, with respect to the Schauder basis $\{e_i\}$, and if there exists $\bar{n} \in N$ such that $n_0 (H_t) \leqslant \bar{n}$, for all $t \in [0, 1]$, then Deg $(I\text{-}H_t, B_r, 0)$ is independent of $t \in [0, 1]$.

Proof : For all $n \geqslant \bar{n}$, H_n: $B_r^n \times [0, 1] \rightarrow X_n$ is a homotopy such that $0 \notin (\cdot\,, t))(\partial B_r^n)$ for $0 \leqslant t \leqslant 1$. By the homotopy property of the Cellina-Lasota degree, we have that Deg $(I\text{-}H_n (\cdot\,, t), B_r^n, 0)$ is independent of $t \in [0, 1]$. Hence also Deg $(I\text{-}H_t, B_r, 0)$ is independent of $t \in [0, 1]$.

Theorem 2. 3 (Excision property)

Let $T: B_r \to X$ be w. u. s. c., with convex values. If T satisfies the condition (A) on ∂S_λ, with respect to the Schauder basis $\{e_i\}$, for any λ with $r \leqslant \lambda \leqslant R$, then $Deg\,(I-T, B_r, 0) = Deg\,(I-T, B_R, 0)$.

Proof : We have that, for all $n \geqslant n_0$, $\deg(I-T_n, B^n_r, 0) = \deg(I-T, B^n_R, 0)$

since $0 \notin (I-T_n)\,(B^n_R \setminus \overset{\circ}{B}{}^n_r)$, where $\overset{\circ}{B}{}^n_r = B^n_r \setminus \partial B^n_r$.

Theorem 2. 4. (Borsuk type theorem).

Let $T : B_r \to X$ be w. u. s. c., with convex values If T satisfies the condition (A), with respect to the Schauder basis $\{e_i\}$, and is odd on ∂B_r, then $Deg(I-T, B_r, 0)$ is odd (i. e., $2m \notin Deg(I-T, B_r, 0)$ for any integer m). In particular, $\{0\} \neq Deg(I-T, B_r, 0)$ so that the equation $x \in Tx$ has a solution in B_r.

Proof : For all $n \in N$, $T_n : B^n_r \to X_n$ is an *u. s. c.* map, with compact

and convex values, which is odd on ∂B^n_r. Hence, for all $n \geqslant n_0$,

$\deg(I-T_n, B^n_r, 0)$ is odd. In fact, the homotopy

$$H_n(x, t) = [1/(1+t)]\,(I-T_n)\,x + [t/(1+t)]\,(-I+T_n)\,x$$

is admissible in the sense of [3], hence

$$\deg\,(H_n(\cdot\,, 0)\ B^n_r, 0) = \deg\,(H_n(\cdot, 1), B^n_r, 0)$$

and there exists $\bar{n} \in N$ such that $n\,(\lambda) \leqslant \bar{n}$, for all $\lambda \in [r, R]$, where $H_n(\cdot\,,1)$ is an odd map on ∂B^n_r. The fact that deg $(H_n(\cdot,1), B^n_r, 0)$ is an odd integer can be shown by constructing a sequence (see [5] , Lemma 2.3) of singlevalued continuous odd maps

$f^n_k : B^n_r \to \overline{\text{co}}\, R(H_n(\cdot\,,1))$, where $\overline{\text{co}}\, R(H_n(\cdot\,,1))$ is the closed convex hull of the range $R(H_n(\cdot\,,1))$ of the map $H_n(\cdot\,,1)$, converging to $H_n(\cdot\,,1)$ in the sense of [3]. Consequently, Deg $(I-T, B_r, 0)$ contains only odd integers .

Theorem 2.5. (Boundary value dependence).

Let *T, $L : B_r \to X$ be two w. u. s. c. maps, with convex values, which satisfy the condition (A), with respect to the Schauder basis $\{e_i\}$. If $Tx=Lx$ for all $X \in \partial B_r$, then* Deg $(I-T, B_r, 0)=$ Deg $(I-L, B_r, 0)$.

Proof : For all $n \in N$ and for all $x \in \partial B^n_r$, $T_n x = L_n x$. For all $n \geqslant \bar{n}$

$=\max\{n_o(T), n_o(L)\}$, deg $(I-T_n, B^n_r, 0)$ and deg $(I-L_n, B^n_r, 0)$ are well defined. Hence the homotopy

$$H_n(x,t) = (1-t)(I-T_n)x + t(I-L_n)x$$

is admissible in the sense of [3] for all $n \geqslant \bar{n}$. Therefore

$$\deg(I-T_n, B^n_r, 0) = \deg(I-L_n, B^n_r, 0) \text{ for all } n \geqslant \bar{n},$$

and hence Deg $(I-T, B_r, 0)=$ Deg $(I-L, B_r, 0)$.

3. A SPECIAL CLASS OF MULTIVALUED MAPS

In this section, let H denote a real separable Hilbert space, We give a sufficient condition for a map T to satisfy condition (A) with respect to every orthonormal basis $\{e_i\}$ in H. To this and, we first prove following

Proposition 3. 1.

For a map $T : B_r \to H$ the following assertions are equivalent

(i) *The map T satisfies the condition (B), with respect to every orthonormal basis $\{e_i\}$.*

(ii) *For all $x \in \partial B_r$ and $y \in Tx$ we have $(y, x) \neq \| x \|^2$.*

Proof : (i) $\Rightarrow$ (ii). Let $x \in \partial B_r$ and $\{e_i\}$ be an orthonormal basis of H, where $e_1 = x/r$. By assumption, the map T satisfies the condition (B), with respect to the orthonormal basis $\{e_i\}$, and $x \in \partial_1 B_r = \{x \in \partial B_r ;\ (x, e_1)^2 = r^2\}$. Hence $(y, x) \neq \| x \|^2$ for each $y \in Tx$.

(ii) $\Rightarrow$ (i). Let $\{e_i\}$ be an arbitrary orthonormal basis of H. If T did not satisfy condition (B) with respect to $\{e_i\}$, then for all $n \in N$ we could find $x \in \partial_n B_r$ and $y \in Tx$ such that $(y, e_j) = (x, e_i)$ for $j = 1, \ldots, n$, hence

$$(y, x) = \| x \|^2, \text{ contradicting (ii).}$$

Theorem 3. 1.

If the map $T : B_r \to H$ is bounded and satisfies the following condition :

$$\text{(b)} \begin{cases} \text{There exists a real number } k > 0 \text{ and an element } a \in H \text{ such} \\ \text{that, for all } x \in \partial B_r \text{ and } y \in Tx, \text{ one of the following} \\ \text{conditions hold :} \\ \text{(I) } (x, y) \neq \| x \|^2 ; \\ \text{(II) } |(a, x-y)| \geqslant k ; \end{cases}$$

then the map T satisfies the condition (A), with respect to every

Orthonormal basis $\{e_i\}$ *in* H.

Proof : Let $\{e_i\}$ be an orthonormal basis in H. Since T is bounded, there exists $n_0 \in N$ such that

$$\left| \sum_{j=n_0}^{\infty} (a, e_j)(y, e_j) \right| < k \qquad \text{for all } y \in TB_r,$$

Now, if the map T did not satisfy the condition (A), with respect to $\{e_i\}$, then for all $n \in N$ we could find $x \in \partial_n B_r$ and $y \in Tx$ such that

$$(x, e_j)=(y, e_j) \quad \text{for } j=1,\dots,n.$$

If x verifies condition I) of (b), this contradicts Proposition 3. 1. On the other hand, if x verifies condition II) of (b), this gives a contradiction for $n \geqslant n_0$, since

$$k \leqslant |(a, x-y)| = \left| \sum_{j=m+1}^{\infty} (a, e_j)(y, e_j) \right| < k .$$

Remark 3. 1.

If $T : B_r \rightarrow H$ is a w. u. s. c. map which satisfies the condition (b) of Theorem 3. 1, then degree of Definition 3.1 is well defined with respect to every orthonormal basis [e_i] in H.

Remark 3. 2.

The class of singlevalued weakly continuous maps which satisfies the condition (b) of Theorem 3. 1 strictly includes the class for which the Canfora-Pacella degree theory applies (see [2] and [6]).

At last we give a fixed point theorem .

Theorem 3. 2.

Let $T : S \to H$ be w. u. s. c. . Suppose that there exists a number $\bar{n}$ such that , for all $n \geqslant \bar{n}$, the condition $(y, x) \leqslant \| x \|^2$ holds for any $x \in \partial_n B_r$ and $y \in Tx$. Then T has a fixed point in B_r .

Proof . We consider the homotopy $H(t, x) = t(I-T)x$. Then the proof of the theorem follows from the proposition 1·2.

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